Geometric Sequences and Series

These lessons offer a number of interesting problems to explore geometric sequences and series. Their main strength is the use of an algorithmic rather than formulaic approach to finding the sum of geometric series.

The lessons and these notes are slightly edited from *Algebra: Themes, Tools, Concepts*, by Anita Wah and Henri Picciotto. More information on this book can be found at:

http://www.picciotto.org/math-ed/attc.html

1. Sequences as Functions

Core Sequence: 1-13

Suitable for homework: 5-13

Useful for Assessment: 13

What this Lesson is About:

- Preview of geometric sequences
- Review of arithmetic sequences
- Linear, quadratic, and exponential functions and their graphs
- Preview of slope

This assignment is intended to focus students' attention on a functional interpretation of sequences. In particular, students should see that arithmetic sequences correspond to linear functions.

The mystery sequences in #11 are quadratic. The numerical pattern that relates consecutive terms in such sequences can be expressed in various ways. One way is to state that the differences between consecutive terms form an arithmetic sequence.

The common difference is visible in the graphs of arithmetic sequences as the "rise" between consecutive points. This previews or reviews slope.

2. Sums of Geometric Sequences

Core Sequence: 3-10

Useful for Assessment: 8, 10

What this Lesson is About:

- Review of comparing by ratio and comparing by difference
- Review of geometric sequences
- Sums of geometric sequences
- Application of geometric sequences to the distance traveled by a bouncing ball

The Bouncing Ball

The experiment in #1-2 is optional, but it greatly increases students' interest in the lesson. It also serves to review comparing by ratio vs comparing by difference. The result of the experiment should be to observe that (even taking measurement error into account) the ratio is much more constant than the difference.

#3-5 should give students a chance to get some experience with the basic problem of the lesson. They will use their calculators to answer the questions.

In #4, make sure students include the travel in both directions. It is easy to make a calculating mistake in #4b, but students should get answers that are close to each others', since the terms get smaller and smaller. Do not attempt 4b without a calculator!

They should be able to make fairly good guesses for #5, either by using their calculators or by making an estimate based on the observation that the terms get smaller and smaller. It is not important for the guess to be accurate.

3. Finding the Sum

Core Sequence: 1-8

Suitable for homework: 6-8

Useful for Assessment: 8, 10-11, 15, 21-22

What this Lesson is About:
- Review of comparing by ratio and comparing by difference
- Review of geometric sequences
- Sums of geometric sequences
- Application of geometric sequences to the distance traveled by a bouncing ball

The terms *geometric sequence* and *common ratio* were introduced in Lesson 1.

The point of this lesson is not to introduce the formula for the sum. There is more gained at this level by students using algebraic manipulation when calculating the sum, than "efficiently" calculating the sum with a formula. Students at this level do not have the maturity to keep in mind both the formula and its derivation. Moreover, it is easy to apply the formula incorrectly because of not remembering whether to use n-1, or n, or n+1 in a given problem. Finally, formula-wielding students often confuse the formula for the n^{th} term with the formula for the sum.

(Any student who is actually conceptually ready to use the formula will be able to do it after this lesson.)

Note that each ball bouncing problem includes the calculation of two sums of geometric sequences, with the second term of one being the first term of the other. This forces students to keep alert and to remember what they are doing.

You may want to introduce the multiply-subtract-solve technique at the chalkboard, before letting students work on #1-5.

4. Decimals and Fractions

Core Sequence: 1-11

Suitable for homework: 8-11

Useful for Assessment: 5, 9

What this Lesson is About:
- Review of converting fractions to decimals
- Converting decimals to fractions

This lesson is mostly an application to decimals-fractions conversion of the previous lesson on sums of geometric sequences .

Writing Fractions as Decimals

#1-2 can be solved with the help of a calculator, however students should be reminded that the limited display of the calculator is not the most appropriate place to study the questions we address in this section, since the decimal expansion may terminate or repeat beyond the level of accuracy allowed by the calculator. The key idea is that remainders whose prime factors (in lowest terms) are exclusively 2 and 5 will terminate. All others will repeat.

#3-4 require the use of long division. The basic idea is that only so many remainders are possible. When they have all been used, the expansion must terminate or repeat. A full answer to #3 is difficult, but some partial answers are:
- if the denominator is d, the length of the repeating string must be < d.
- in fact, the length of the repeating string is often a factor of (d-1).

#5 is an opportunity to distinguish between a decimal approximation and a decimal representation for a given fraction.

Writing Decimals as Fractions

You should probably demonstrate on the chalkboard how to apply the multiply-subtract-solve technique to the conversion of repeating decimals to fractions.

5. More Practice

Rational Numbers

can be answered by using the multiply-subtract-solve technique. However, do not do not be concerned if some students do not find the argument convincing. A real understanding of limits requires more mathematical maturity.

Bounce Ratios

This section will allow students to practice what they have learned, but also to think about the difference between a common ratio r such that $r < 1$, and one such that $r \geq 1$.

You do not need to formally define convergence. At this level, all students need to realize is that the higher the exponent of r (if $r<1$), the smaller the corresponding term.

There will be more applications of sums of geometric sequences in the following lessons. In addition, you can go back to the problems in Chapter 2, Lesson 5 and Chapter 7, Lesson 8.