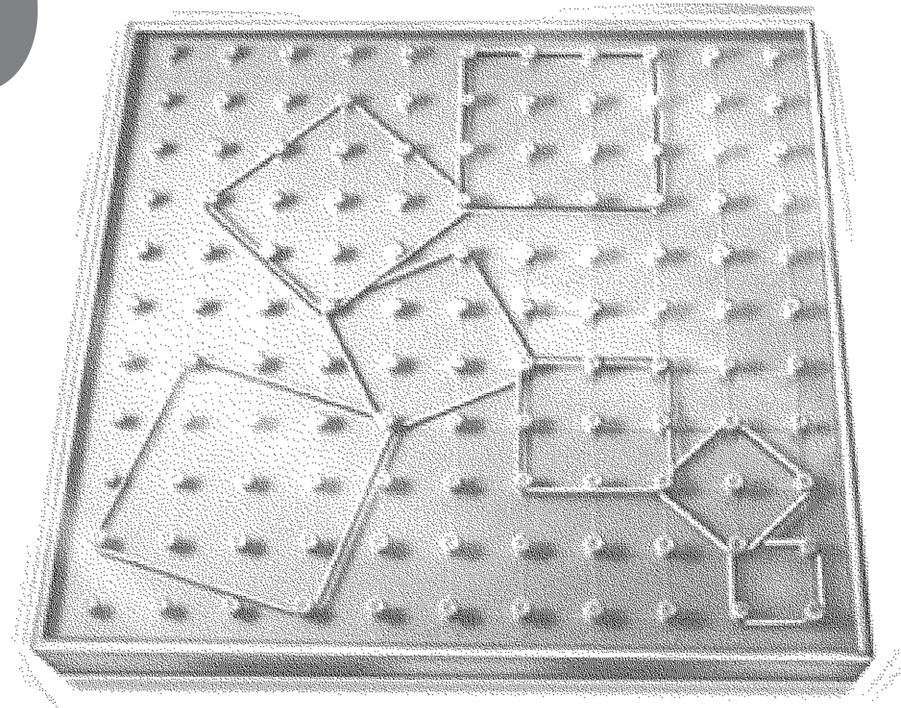
8 Perimeter and Area



This section starts a discussion of perimeter and area, which will be continued throughout the rest of this book. Contexts involving square roots and the Pythagorean theorem will be taken up in Section 9. Contexts involving similarity and scaling will come up in Section 10. And finally, contexts involving trigonometry will be taken up in Section 11.

Some of this material is explored (with an emphasis on connections to algebra) in the textbook I coauthored: *Algebra: Themes, Tools, Concepts* (Creative Publications, 1994).

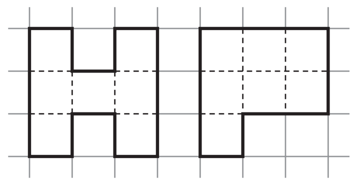
See page 208 for teacher notes to this section.

LAB 8.1
Polyomino Perimeter and Area

Name(s) ____________________

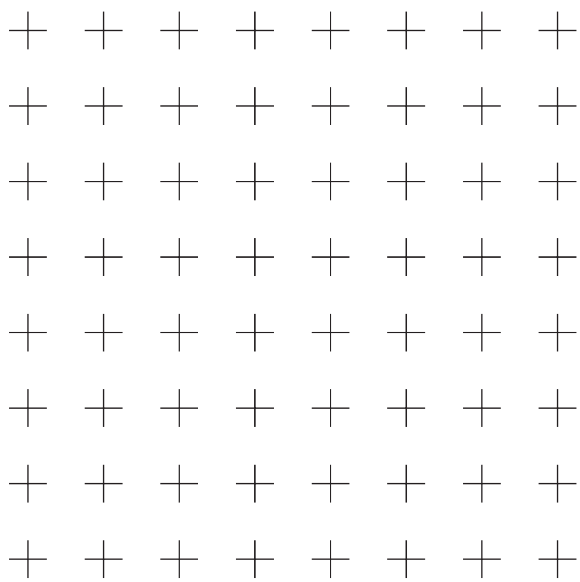
■ **Equipment:** 1-Centimeter Grid Paper, interlocking cubes

A polyomino is a graph paper figure whose outline follows the graph paper lines and never crosses itself. We will only consider polyominoes with no holes, such as these:



1. What are the area and perimeter of each polyomino shown above?

2. Find a polyomino with the same area as the ones above, but with a different perimeter. Sketch it in the grid below.



3. Experiment on graph paper or with the help of your interlocking cubes and fill out the following table.

Area	Minimum perimeter	Maximum perimeter
1		
2		
3	8	8
4		
5	10	12
6		
7		
8		
9		
10		
11		
12		
13		

Area	Minimum perimeter	Maximum perimeter
14		
15		
16		
17		
18		
19		
20		
21		
22		
23		
24		
25		
26		

4. Find a formula for the maximum perimeter, P_{max}, for a given area A.

5. Describe a pattern for the minimum perimeter.

6. What would the minimum and maximum perimeters be for the following areas?

a. 49 min. ________ max. ________
b. 45 min. ________ max. ________
c. 50 min. ________ max. ________
d. 56 min. ________ max. ________

Discussion

A. Is it possible to draw a polyomino with an odd perimeter? Explain how to do it, or why it is impossible.

B. While filling out the table, what was your strategy for finding the maximum and minimum perimeters?

C. Graph minimum and maximum polyomino perimeters with area on the x-axis and perimeter on the y-axis. What does the graph show about the formula and pattern you found in Problems 4 and 5?

D. Find the polyominoes whose area and perimeter are numerically equal.

E. Explain your strategy for answering the questions in Problem 6.

LAB 8.2

Name(s) ______________________

Minimizing Perimeter

■ **Equipment:** 1-Centimeter Grid Paper, the results of Lab 8.1

In this lab, you will try to get a better understanding of the minimum perimeter pattern you discovered in Lab 8.1.

By studying the numbers in the table in Lab 8.1, you will notice that when the area increases by 1, the minimum perimeter stays the same or increases by 2.

1. Thinking about them numerically, what is special about the areas that precede increases of the minimum perimeter? **Hint:** There are two different types of such areas.

2. Circle all perfect square areas in the table in Lab 8.1.

3. Circle all areas that are the products of consecutive whole numbers. For example, circle 6, since $6 = 2 \cdot 3$. (We will call these products *rectangular numbers*.)

Square and rectangular numbers are the key to predicting the minimum perimeter for any area. For example, consider the square number 900.

4. What are the minimum perimeters for the following areas?

a. 900

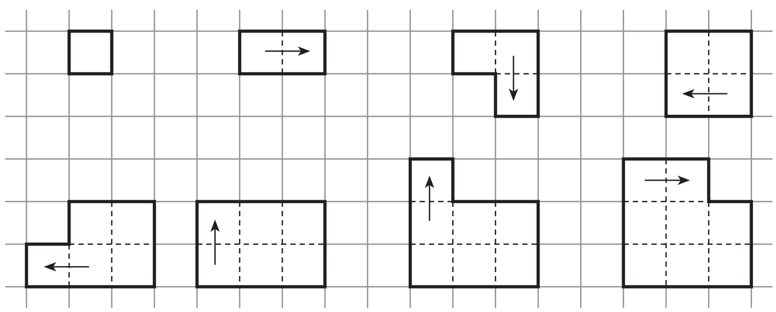
b. 901

c. 895

5. What is the greatest rectangular number that is less than 900? What is the least rectangular number that is greater than 900?

6. What are the minimum perimeters for the following areas?
   - **a.** 925
   - **b.** 935
   - **c.** 865

In order to understand the geometry behind the special role of square and rectangular numbers, you can draw consecutive minimum-area polyominoes by spiraling around an initial square, as shown in the figure below.



7. Continue the process shown in the figure. Describe situations when the perimeter stays the same, and when it changes.

Discussion

A. Explain why square and rectangular numbers are important in understanding this lab.

B. Explain, with the help of figures, why the perimeter of a polyomino whose area is a little less than a given square or rectangular number is the same as the perimeter of the square or rectangle.

C. What are the minimum perimeters of polyominoes with the following areas?
   - **a.** n^2
   - **b.** $n(n + 1)$

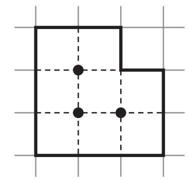
LAB 8.3
A Formula for Polyomino Perimeter

Name(s) ________________________

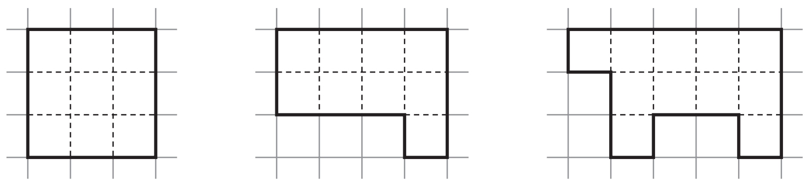
■ **Equipment:** 1-Centimeter Grid Paper, interlocking cubes

It is possible to find a formula for the perimeter of a polyomino as a function of its area and of one other variable: the number of *inside dots*.

Inside dots are the points of intersection of grid lines in the interior of the polyomino. For example, the polyomino at right has three inside dots.



1. What are the perimeter, the area, and the number of inside dots for the figures below?

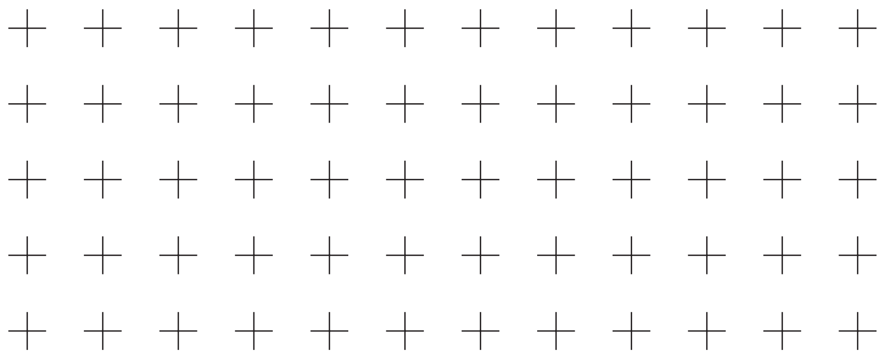


Experiment on grid paper or with your interlocking cubes to find the formula. Keep track of your experiments in the table. Problems 2–5 can help you organize your research.

Perimeter	Area	Inside dots

Perimeter	Area	Inside dots

2. Create a series of figures, all with the same area but with different numbers of inside dots. Keep a record of the area, perimeter, and number of inside dots in the table.
3. Every time you add an inside dot, how are you changing the perimeter?
4. Create a series of figures, all with the same number of inside dots but with different areas. Keep a record of the area, perimeter, and inside dots in the table.
5. Every time you add one square unit to the area, how are you changing the perimeter?
6. Find a formula for the perimeter P of a polyomino if its area is A and it has i inside dots.
7. **Puzzle:** Using interlocking cubes, make all possible polyominoes with perimeter 10. Use only one color per polyomino. Arrange them in a rectangle and record your solution below.



Discussion

A. If two polyominoes have the same area and the same perimeter, what can you say about the number of inside dots?

B. If two polyominoes have the same area, but the perimeter of one is greater than the perimeter of the other, what can you say about their respective numbers of inside dots?

C. What is the least number of inside dots for a given area?

D. What is the greatest area for a given number of inside dots?

E. What can you say about the number of inside dots for a square of side n?

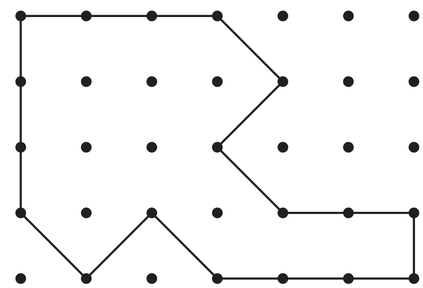
F. What can you say about the number of inside dots for a rectangle of dimensions n and $n + 1$?

LAB 8.4
Geoboard Area

Name(s) ____________________

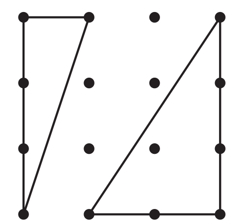
■ **Equipment:** Geoboard, dot paper

The area of the geoboard figure at right is 15.



1. Find other geoboard figures with area 15. The boundaries of the figures need not be horizontal or vertical. Find figures that are different from the ones your neighbors find. Record your solutions on dot paper.

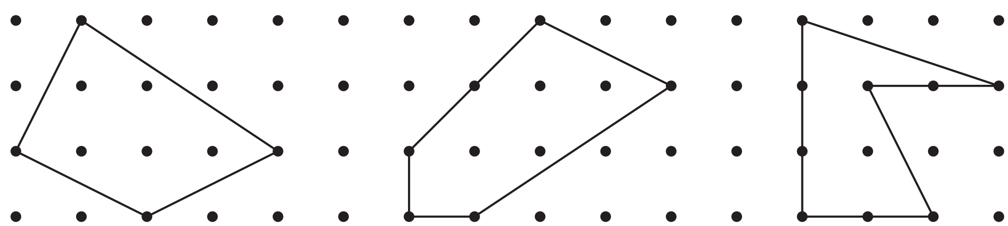
It is easiest to find areas of geoboard rectangles with horizontal and vertical sides. The next easiest figures are the "easy" right triangles, such as the two shown at right.



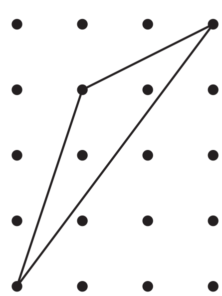
2. Find the areas of these triangles.

If you can find the area of easy right triangles, you can find the area of any geoboard figure!

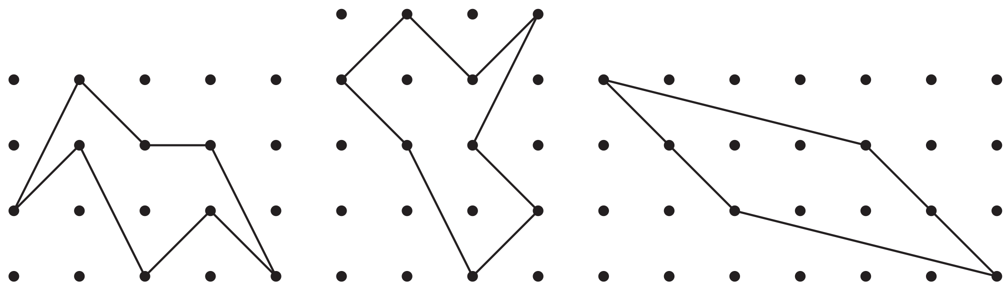
3. Find the areas of the figures below.



4. Find the area of the figure below. (It may be more difficult than Problem 3. **Hint:** Use subtraction.)



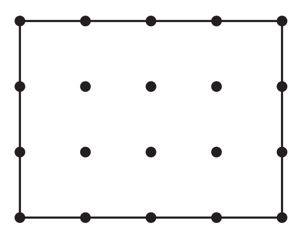
5. Find the areas of the figures below.



6. How many noncongruent geoboard triangles are there with area 8? Limit yourself to triangles that can be shown on an 11×11 geoboard and have a horizontal base. Record your findings on dot paper.

7. **Puzzle:** Find the geoboard figure with the smallest area in each of these categories.
   - **a.** Acute triangle
   - **b.** Obtuse triangle
   - **c.** Right triangle
   - **d.** Square
   - **e.** Rhombus (not square)
   - **f.** Rectangle (not square)
   - **g.** Kite
   - **h.** Trapezoid
   - **i.** Parallelogram

Discussion



A. A common mistake in finding geoboard areas is to overestimate the sides of rectangles by 1 (for example, thinking that the rectangle at right is 4×5). What might cause this mistake?

B. Explain, with illustrated examples, how the following operations may be used in finding the area of a geoboard figure: division by 2; addition; subtraction.

C. What happens to the area of a triangle if you keep its base constant and move the third vertex in a direction parallel to the base? Explain, using geoboard or dot paper figures.

D. Use geoboard figures to demonstrate the area formulas for various quadrilaterals.

LAB 8.5
Geoboard Squares

Name(s) ______________________

■ **Equipment:** Geoboard, dot paper

1. There are 33 different-size squares on an 11 × 11 geoboard. With the help of your neighbors, do the following.

a. Find all the squares.

b. Sketch each square on dot paper, indicating its area and the length of its side.

Discussion

A. How can you make sure that two sides of a geoboard square really form a right angle?

B. How can you organize your search so as to make sure you find all the squares?

C. Is it possible to find squares that have the same area, but different orientations?

D. In the figure at right, find the following in terms of a and b.

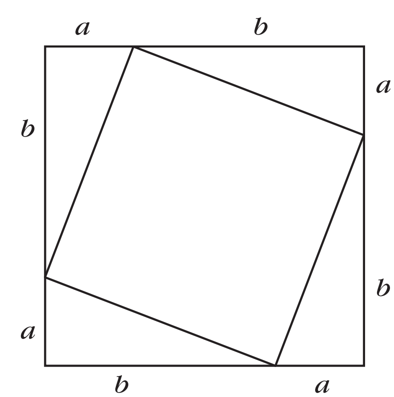
a. The side of the outside square

b. The area of the outside square

c. The area of each triangle

d. The area of the inside square

e. The side of the inside square

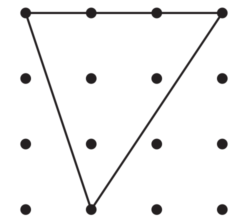


LAB 8.6

Name(s) ______________________

Pick's Formula

■ **Equipment:** Geoboard, dot paper

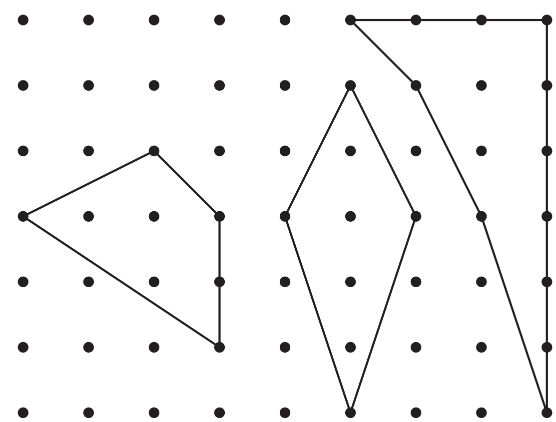


It is possible to find a formula for geoboard area as a function of boundary dots and inside dots.

For example, the geoboard figure at right has three inside dots and five boundary dots.

1. What is the area of the figure above?

2. What are the area, the number of inside dots, and the number of boundary dots for each of the figures at right?



Experiment on your geoboard or on dot paper to find the formula. Keep track of your experiments in the table below. Problems 3–6 can help you organize your research.

Inside dots	Bound. dots	Area

Inside dots	Bound. dots	Area

Inside dots	Bound. dots	Area

Name(s) ____________________

3. Create a series of geoboard figures, all with the same number of boundary dots but with different numbers of inside dots. Keep a record of the area, boundary dots, and inside dots in the table.
4. Every time you add an inside dot, how are you changing the area?
5. Create a series of figures, all with the same number of inside dots but with different numbers of boundary dots. Keep a record of the inside dots, boundary dots, and area in the table.
6. Every time you add 1 to the number of boundary dots, how are you changing the area?
7. Find a formula for the area of a geoboard figure if it has b boundary dots and i inside dots.

Discussion

A. What is the greatest area for a given number of inside dots?

B. What is the least area for a given number of boundary dots?

C. How is the formula you found in this lab related to the one you found in Lab 8.3 (A Formula for Polyomino Perimeter)?