Isometries: Teacher Notes

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Acknowledgments

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Introduction

A transformation of the plane is a one-to-one function that takes as an input any point in the plane, and gives as an output a unique point. The Common Core State Standards (CCSS) give a foundational role to *dilations*, which stretch figures proportionally, and *isometries*, the transformations that preserve distance.

In middle school and at the start of high school geometry, students get acquainted with dilations and the basic isometries: translations, rotations, and reflections. They also develop an understanding of the concepts of congruence and similarity, which are defined as the relationship between two figures if one is obtained from the other by a sequence of these transformations.

There are many reasons why transformational geometry is becoming a big part of the school curriculum. Pedagogically, it is more accessible than many traditional geometric concepts. Mathematically, it provides better connections between algebra and geometry. And in terms of its applications, it is the basis of all computer graphics, from moving windows on a screen, to video games, to animated films.

Needed Materials

rulers
Geometry Labs Templates (available from Nasco)
Geomirrors (available from Didax) or Miras
graph paper (ideally, the kind where every fifth line is heavier)
triangle paper
patty paper

GeoGebra

Where it makes sense, I will reference activities that can be done with GeoGebra, in some cases using files you can download from <www.MathEducationPage.org>.

Many of those activities can be carried out with Geometer’s Sketchpad or Cabri. If students don’t yet have access to this technology, those activities are worth doing as a teacher-led demonstration and discussion.

Lessons

1. Isometries

You might start with a discussion of the applications of transformational geometry to animation and video games, perhaps by showing brief examples and discussing what motion students see. Of course animated movies and contemporary video games involve transformations in three dimensions, and at this level students will just be working in two dimensions. 2D examples of animation include scrolling on a computer screen or in movie credits, and old-style video games such as Tetris or Pac-Man. (There are versions of these two games available free online.)

Follow that up with a GeoGebra discussion. (You will need to download the demonstration files from http://www.mathedpage.org/transformations/isometries.) The first three files demonstrate the three basic isometries. The next three are “mystery isometries”. You could use them now, or save them for the end of the lesson. Ask the students to guess which basic isometry is represented in each case, and perhaps have them vote on it. Follow up with questions like “where is the reflection line?”, “which way does the vector point?”, and “where is the center of rotation?” (The last question is quite challenging for beginners, so after a bit of discussion, press the “cheat” button!)

In particular, you can use the Reflection GeoGebra file to point out that the line of reflection is the perpendicular bisector of the segment joining a point to its image. If your students are not yet familiar with the perpendicular bisector, one way to introduce it is to take students outside the classroom, perhaps to a playground or gym. Choose two students to be “point A” and “point B”. Tell the others to stand closer to A. Then closer to B. Then at the same distance from A and B. They will find themselves on a line. Ask for a description of the line. (It is perpendicular to the segment AB, and passes through its midpoint.) Tell the class that B is the reflection of A in the line, and that is in fact the mathematical definition of reflection.

Be sure to also demonstrate the isometries kinesthetically in other ways, by showing the image of a physical object, using each of the basic isometries, and then asking the students to do the same.

Then, have them work through the worksheet.

Finally, you can have a closing discussion about the mystery GeoGebra isometries if you haven’t yet done that.

Note that even though isometries are also called “rigid motions”, the idea of motion is only to help us understand what is going on. Mathematically, we are just concerned with the pre-image and the image, neither of which is moving.

(You can complement all this with a demonstration of the three basic isometries on a document camera, &/or using transparencies, or patty paper, or some other way. The main point here is to get familiar with the basic isometries and the accompanying vocabulary.)

2. Isometries Practice

You should precede each section with a step-by-step demonstration of the process on an overhead projector or document camera. (While it is not usually a good idea to start a lesson with an explanation of how to solve problems, it would make no sense to expect students to invent the definitions of the transformations. The teacher must introduce these. Students will be asked to solve problems in subsequent lessons, once they are familiar with the meaning of the transformations.)

Before starting the worksheet, you should give students a chance to familiarize themselves with the tools. Let them make whatever designs they want using the template. Explain how to use the see-through mirrors, as that is far from obvious to a beginner. (If you want to see the reflection of a drawing, you need to look from the side the drawing is on.)

After the first example of each type, encourage students to choose their own vectors, angles, pre-images, and so on. This will give them more ownership of the activity and deeper understanding. In particular, make sure their vectors do not all point up and to the right. Encourage the use of asymmetric and not overly simple pre-images, to make the exercise more fruitful and interesting.

To make clear the relationship between image and pre-image, you can ask students to add marks in corners of the pre-image, and add the image of the marks at the corresponding location in the image.

Welcome student attempts at making this look good, for example by using color, as long as it does not take an inordinate amount of time.

> GeoGebra: if your students have access to interactive geometry software, they can follow up this activity by practicing the basic isometries on the computer.

3. No Template, No Mirror!

And also no patty paper, no transparencies.

The idea here is to use the pre-existing lines on graph paper and triangle paper as guides to carry out the basic isometries. Again, let students create their own vectors, etc. as long as they follow the guidelines on the worksheet.

Note that the formal definition of *lattice point* is a point with integer coordinates. In this activity, we will use it to denote the intersection of the lines on the graph paper and triangle paper.

The assignment is more accessible if the pre-image follows graph paper lines. However there is no reason to prevent students from using diagonals or even curves if they want to do something fancier and are up to the challenge.

For the section on rotation, if students are not familiar with the needed angles, help them by demonstrating with a simple example. Do not spend a lot of time explaining how you know those angles, as this could derail the whole lesson for the benefit of a few students. Instead, be very explicit in your help. Doing this work will provide good practice to familiarize students with these famous angles. (Of course, if most students need this help, you may need to spend some time explaining how these angles are derived from 360°. See the 7^{th} grade **Angles** packet.)

For the section on reflection, you can allow the use of the see-through mirrors, but only to check the correctness of the drawings.

4. Isometries Puzzles

In this activity, the situation is reversed: the pre-image and the image are given. The challenge is to find which isometry will get us from one to the other.

Follow this up with an activity on a document camera.

1. Use the template to draw a polygon somewhere on a piece of graph paper, making sure the template is aligned with the graph paper. (The purpose of the graph paper is simply to be able to move the template without tilting it when solving the puzzle later on.)

2. Draw the same polygon somewhere else on the graph paper. (Or, to put it more formally, a congruent polygon.)

The challenge will be to figure out how to get from the first to the second, using a sequence of basic isometries. There will always be many ways to do it.

One strategy that will always work is to choose a vertex, and translate the original shape so that that vertex lands on its image. If the whole figure is superposed on the image, you're done. If not, you may be able to rotate it around that vertex so it is superposed on the image. If that is impossible, you may need to reflect it across a side, and then rotate.

5. Some Things Don't Change

It would make sense for students to work on this in groups of three, so that they can each take responsibility for one of the basic isometries for each question. However they should switch that responsibility around.

The purpose of this activity is for students to find out what changes and what stays the same after an isometry. Before doing the activity on the worksheet, you should solicit predictions about the items to be investigated. After the work they have done in the previous lessons, students will have no trouble making correct predictions for several of these. Do not insist on them doing the experiment in cases where the answers are obvious, but their reports need to be complete.

The idea of orientation is subtle. This is a very specific meaning of "orientation", which is different from the one used in everyday language, and even in some mathematical contexts. A triangle does not intrinsically have an orientation. However if we number its vertices, or label them in alphabetical order, then we can say that its orientation is clockwise or counterclockwise, by following the vertices in order. The orientation of a figure is reversed by a reflection, but it is preserved by a translation or rotation.

Zalman Usiskin suggests a clever way to confirm that rotations preserve orientation: a 1° rotation clearly does not change the orientation. But a rotation by, say, 180° can be done as a succession of 1° rotations! (Of course, it's still a good idea to verify that by checking whether A'B'C' still go around the same way as ABC.)

Note that students often misunderstand the question posed in #2.c. We are not asking whether the image of a line is parallel to the original. We are asking whether the images of two parallel lines are parallel to each other.

> GeoGebra: this activity can be done on the computer using interactive geometry software.

6. Translations and Coordinates

This is a look at what happens to coordinates under translations in the coordinate plane. Along with the next two lessons, it will provide a good way to review the unit.

7. Reflections and Coordinates

This is a look at what happens when a figure is reflected in one of the axes.

Note that #4 is a follow-up to the challenge in #7 of **No Template, No Mirror!** You might want to have students build directly on any work they or their classmates did on that problem.

#5 is even more challenging. Encourage students to keep their discoveries organized in tables

8. Rotations and Coordinates

This is a look at 90° and 180° rotations around the origin. Students will need some support, as this will be difficult for many. If things go well, a 270° rotation could provide an extension.

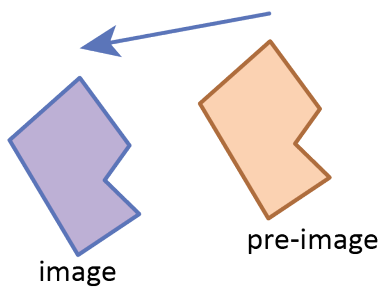
1. Translation, Rotation, and Reflection

We will learn three different ways make an exact copy of a figure in the plane. They are called *translation*, *rotation*, and *reflection*.

The original figure is called the *pre-image*. The figure after the motion is called the *image*.

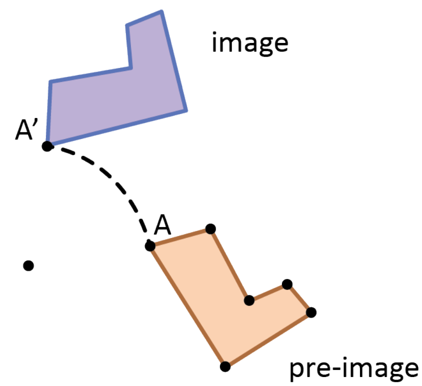
Translations, rotations, and reflections are all examples of *isometries*, motions that do not change an object's size or shape.

Translation



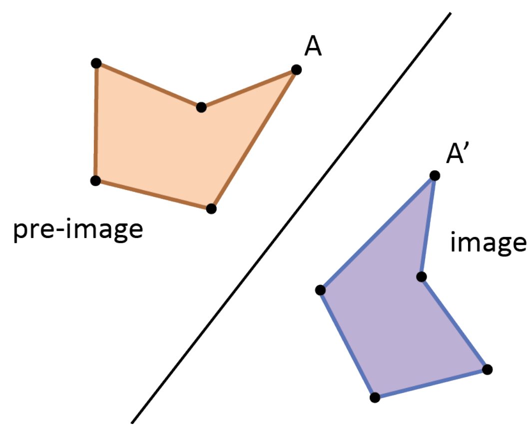
1. In a translation every point of the pre-image is moved by the same distance and in the same direction to form the image. In the figure above, the arrow represents the distance and direction that each point is moved. The arrow is called a *vector*.
   a. In the figure above, write "vector" on the vector.
   b. Choose a vertex on the pre-image and label it A.
      Find the corresponding vertex on the image, and label it A'.
      Use a ruler to draw the vector that connects point A to A'.
   c. Choose another point on the pre-image, and label it B.
      Find the corresponding point on the image, and label it B'.
      Use a ruler to draw the vector that connects point B to B'.
   d. How do the length and direction of the vectors that you have drawn compare to the length and direction of vector already drawn on the diagram?

Rotation



2. This diagram illustrates rotation. In a rotation every point turns by the same angle around a center point, following along a circle.
   a. In the figure above, write "center" next to the *center of rotation*.
   b. Two points A on the pre-image and its image A' are labeled. Draw the angle of rotation, by drawing line segments from the center to the points A and A' .
   c. Choose two other vertices in the pre-image and label them B and C.
   d. Label their images B' and C'.

Reflection

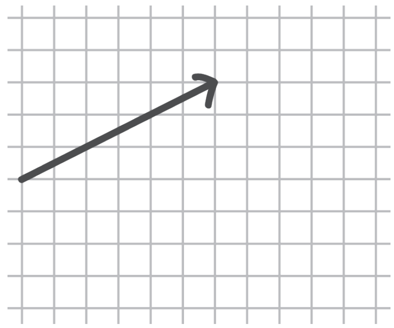


3. In a reflection, every point goes to its mirror image across a line.
   a. Write "reflection line" on the mirror line.
   b. A vertex A and its image A' are labeled. Draw the line segment connecting them.
   c. Mark the segment you drew to show any equal segments and/or right angles.
   d. Choose two other vertices in the pre-image and label them B and C.
   e. Label their images B' and C'.

2. Isometries Practice

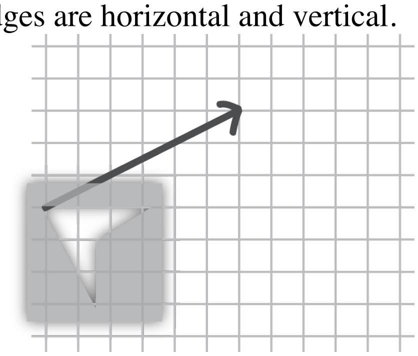
Translation

1. Practice translation on a piece of graph paper, with the help of your template:
   a. Draw a translation vector just like this one.

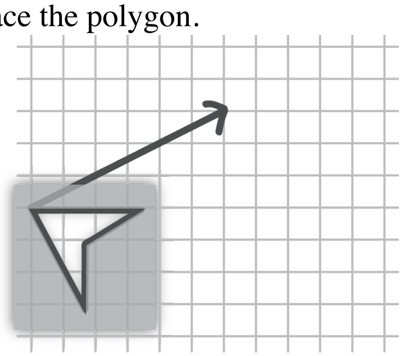


   b. Find this polygon on the template.

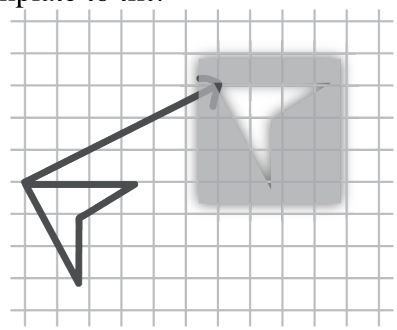
   c. Place your template on the paper so that one vertex of your polygon is at the start of your vector, using the graph paper lines to make sure your template edges are horizontal and vertical.



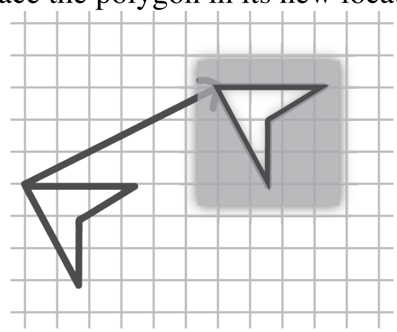
   d. Trace the polygon.



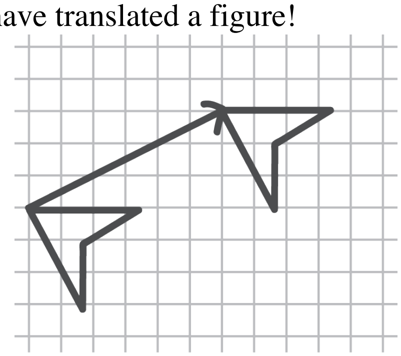
   e. Slide the template so that the same vertex is now at the end of your vector, again using the graph paper lines to make sure your template edges are horizontal and vertical. Do not allow the template to tilt!



   f. Trace the polygon in its new location.



You have translated a figure!

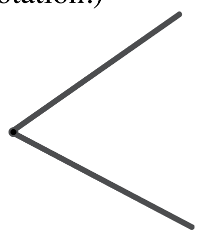


2. Do this a few times, with different vectors and different polygons.

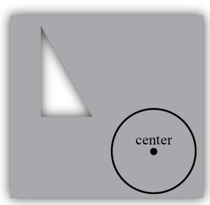
Rotation

3. Practice rotation on a piece of unlined paper, with the help of your template:
   a. Mark a point somewhere on the paper. That will be your center of rotation.

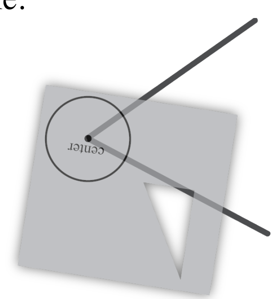
   b. Draw the two sides of the angle of rotation (two rays originating at the center of rotation.)



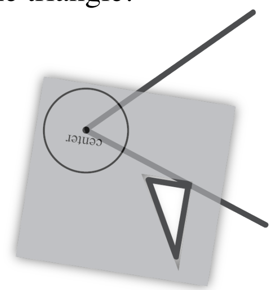
   c. Find this triangle on the template. It is above the circle and to the left.



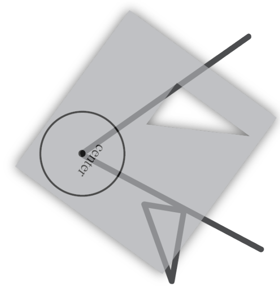
   d. Place your template on the paper so that the center of the template's circle is on the point you marked, and so that one vertex of your triangle is on one side of the angle.



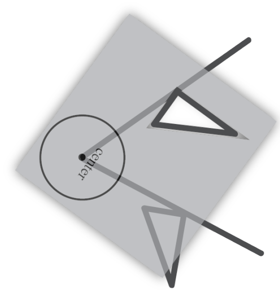
   e. Trace the triangle.



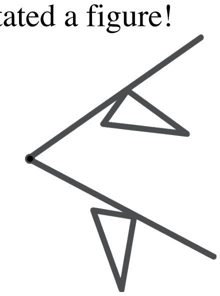
   f. Put your pencil at the center of the template's circle, and rotate the template around that point so that the same vertex is now on the other side of the angle. Do not allow the paper to turn!



   g. Trace the triangle in its new location.



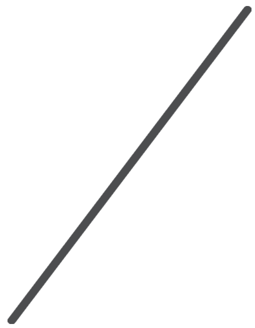
You have rotated a figure!



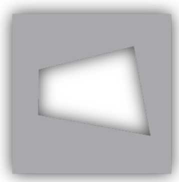
4. Do this a few times with different centers, angles, and polygons.

Reflection

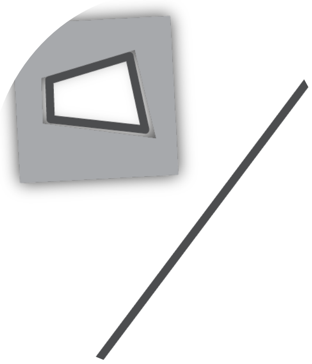
5. Practice reflection on a piece of unlined paper, with the help of your template and a plastic see-through mirror:
   a. Draw the line of reflection.



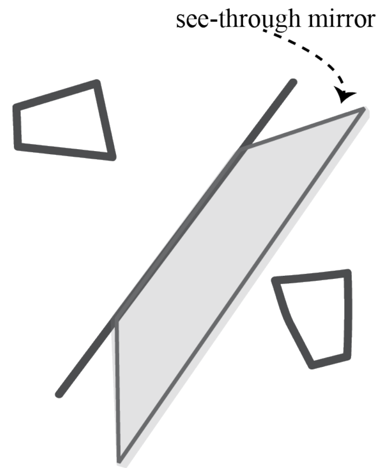
   b. Find this polygon on the template.



   c. Trace it on one side of the mirror line.

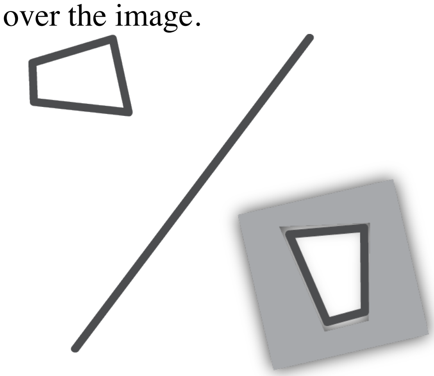


   d. Place the mirror on the mirror line, and draw your polygon's image on the other side of the line, looking at the reflection in the mirror.



You have reflected a figure!

   e. If you want it to look even better, flip the template over, and use it to trace over the image.



6. Do this a few times with different figure and mirror lines. You can even do it without the template: make a simple drawing, and use the mirror to draw its reflection.

3. No Template, No Mirror!

You will now practice isometries first on graph paper, then on triangle paper, without template or mirror.

In this activity, we will define a *lattice point* as a point where the graph paper or triangle paper lines meet.

Translation

1. On graph paper:
   a. Draw a translation vector that starts on one lattice point and ends on another lattice point.
   b. Draw a pre-image, following the graph paper lines.
   c. Draw the image after the translation.

2. Do it again, on triangle paper.

Rotation

3. On graph paper, using a 180° angle of rotation.
   a. Mark a center of rotation on a lattice point, and two rays to show your angle of rotation.
   b. Draw a pre-image, following the graph paper lines.
   c. Draw the image after the rotation.

4. Do it again, using a 90° angle, clockwise.

5. Do it again, on triangle paper, using a 60° angle, counterclockwise.

6. Do it again, on triangle paper, using a 120° angle, clockwise.

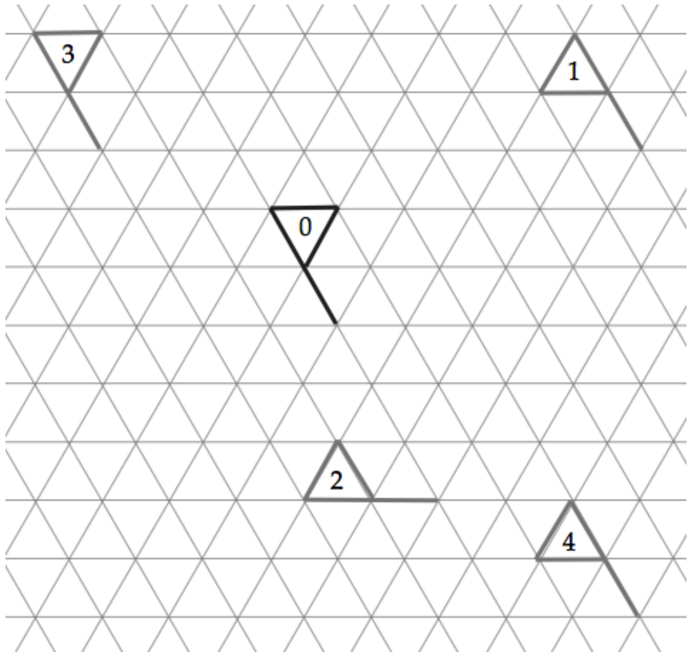
Reflection

7. On graph paper:
   a. Draw a line of reflection, using a graph paper line.
   b. Draw a pre-image, following the graph paper lines.
   c. Draw the image after the reflection.

8. Repeat on triangle paper.

9. **Challenge:** On graph paper, make your reflection line at a 45° angle to the graph paper lines.

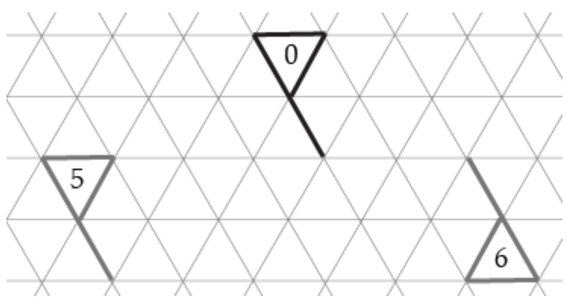
4. Isometries Puzzles



1. Which basic isometry will take flag 0 to each of flags 1, 2, and 3? Draw and label the vector for the translation, the center and angle for the rotation, and the mirror line for the reflection.

2. You will need two or more basic isometries, one after the other, to get from flag 0 to flag 4. Find a way to do it.

Challenges



3. You can get from flag 0 to flag 5 using two reflections, one after the other. Draw the two mirror lines, and the flag after the first reflection.

4. You can get from flag 0 to flag 6 using two reflections, one after the other. Draw the two mirror lines, and the flag after the first reflection. **Hint:** One of the lines is not part of the triangle paper lines.

5. Challenge a classmate! Using graph paper, draw a figure following the graph paper lines. Use two basic isometries, one after the other, to get the figure somewhere else on the page, but do not show any vectors, centers, or mirror lines, and do not show where the figure is after just one transformation. Have a classmate get from your original figure to your final figure using only isometries. (They don't have to use the same ones you did, but they do have to end up exactly in the same place.)

5. Some Things Don't Change

In this activity, you will compare a pre-image to its image after an isometry, and try to figure out what changes, and what stays the same.

Work with other students so as to be able to try different pre-images and different isometries. For example, one student can use a translation, another a rotation, and a third a reflection, and then switch it around for the following experiment. Check each other's work to make sure that your group's conclusions are valid.

Do your work on graph paper, and keep track of what you find out.

For most of these problems, you will find that the same result applies to all three basic isometries. But be careful: in at least one of these problems, that is not the case. If there are differences between the different motions, make a note of them.

Experiments

1. Measurements:
   a. *Angles*: Draw a triangle, and create an image of the triangle using an isometry. Use patty paper to see if the angles in the image are equal to the angles in the pre-image.

   b. *Length* (distance): Draw a line segment and create an image of the segment using an isometry. Use patty paper to see if the length of the image is the same as the length of the pre-image. (Another way to say this is, "see if the distance between the endpoints changed or stayed the same." Or: "see if distance is preserved")

2. Relationships:
   a. *Collinear points*: Choose three points that lie on one straight line (not necessarily a graph paper line.) We say the three points are *collinear*. Create images of the three points using the same isometry for each. Do the images of the points lie on one straight line?

   b. *Orientation*: Choose three points that make a triangle, and label them A, B, C, going around clockwise. Create images of your points using the same isometry for each, and label them A', B', and C'. If you go around A'B'C' in order, are you still going clockwise? (Clockwise and counterclockwise are how we describe the orientation of a triangle once we have labeled its vertices.)

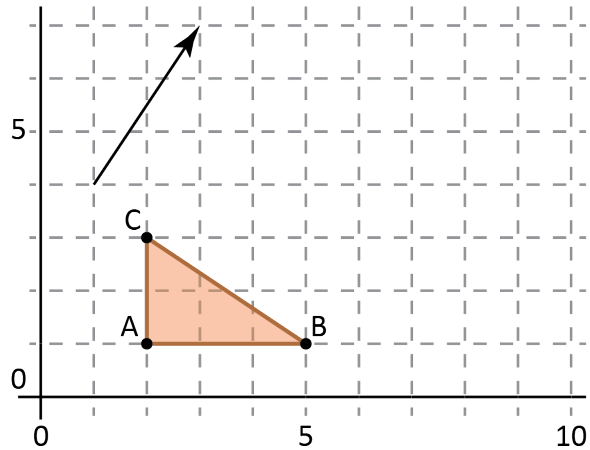
   c. *Parallel lines*: Draw two parallel lines, and create images of the two lines using the same isometry for each line. Are the images of the two lines parallel?

Conclusion

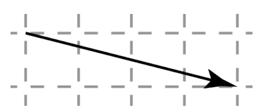
Based on your experiments, discuss with your group what changes and what stays the same after an isometry. Create a report by writing down and illustrating your conclusions on graph paper, making sure you all agree with each other.

Organize your report in three parts: (a) what changes for all isometries? (b) what stays the same?; and (c) what depends on which isometry we are talking about?

6. Translations and Coordinates



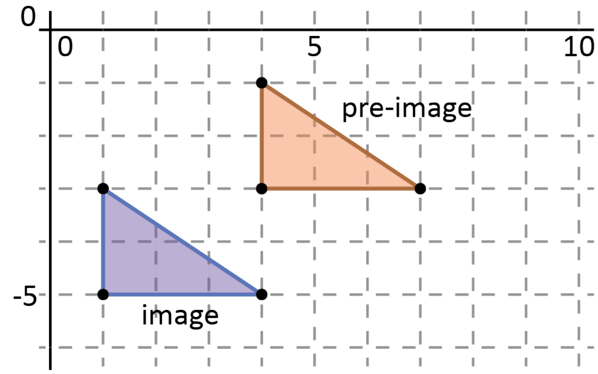
1. Draw the image of triangle ABC after translating it using the vector shown in the figure.
2. Write down the coordinates of A, B, and C, and the coordinates of their images A', B', and C'.
3. What would the coordinates of A', B', and C' be if you had used this vector instead?



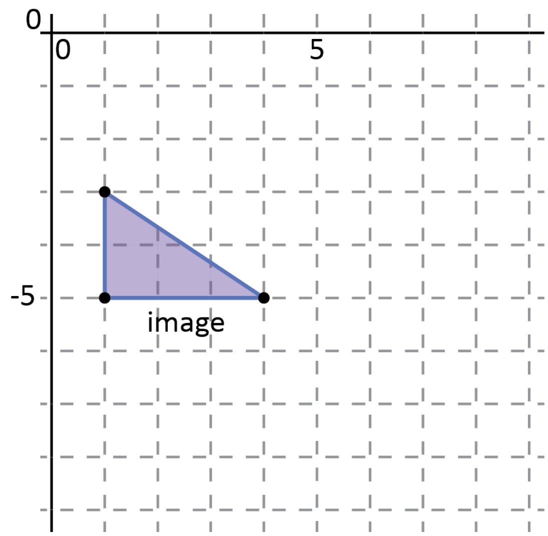
4. Vectors can be described by two numbers: how much *x* changes from the start to the end of the vector, and how much *y* changes. Those numbers are the horizontal and vertical components of the vector.
   a. What are those numbers for the vector in #1?

   b. How about the vector in #3?

5. Here is a pre-image and an image. What are the horizontal and vertical components of the translation vector?

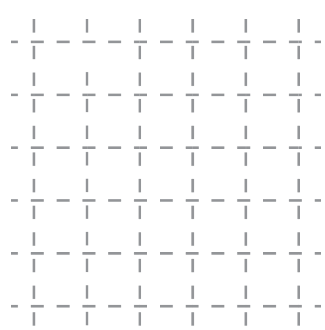


6. Here is an image. The horizontal and vertical components of the translation vector are -3 and 3. Draw the pre-image.

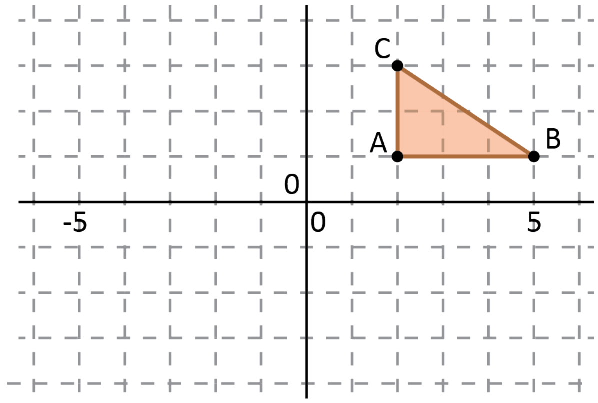


7. **Challenge:** A triangle has vertices (1,2), (3,5), and (1,5). What are the coordinates of the vertices after a translation with a vector whose horizontal and vertical components are -3 and 3?

8. **Challenge:** Think of a point P, with unknown coordinates (x,y). After a translation, the new coordinates are (x–2, y+5). Draw the translation vector:



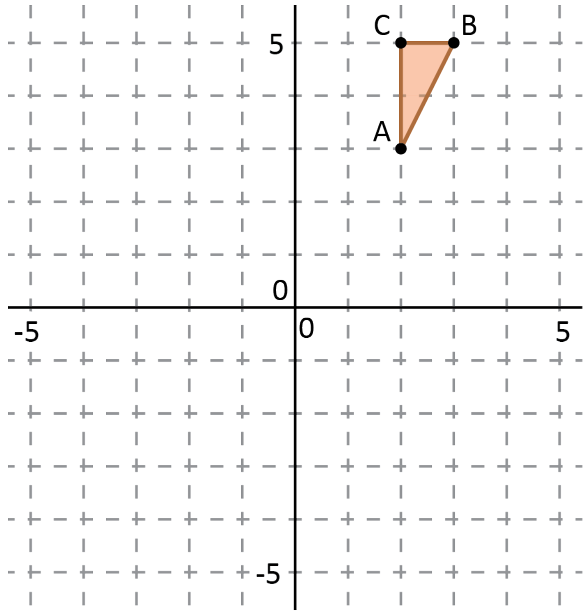
7. Reflections and Coordinates



1. Draw the image of the triangle after reflecting it in the *x*-axis. What happened to the coordinates?

2. Draw the image of the original triangle after reflecting it in the *y*-axis. What happened to the coordinates?

3. Draw the image of the original triangle after reflecting it in the *x*-axis, and then reflecting that image in the *y*-axis. What happened to the coordinates?



4. **Challenge:**
   a. Write down the coordinates of A, B, and C.

   b. Draw the graph of $y = x$.
   c. Reflect the triangle in the line. Write down the coordinates of the vertices.

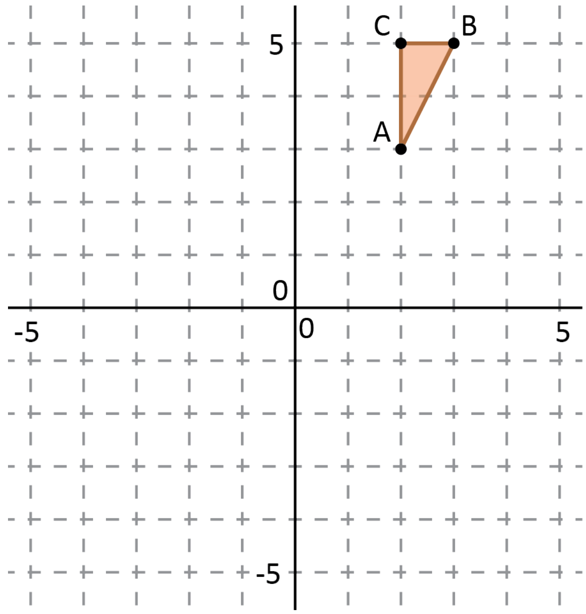
   d. What happens to the coordinates of a point if it is reflected in the line $y = x$?

   e. Draw the graph of $y = -x$.
   f. Reflect the original triangle in the line. Write down the coordinates of the vertices.

   g. What happens to the coordinates of a point if it is reflected in the line $y = -x$?

5. **Challenge:** What would happen to its coordinates if we reflected a point in a vertical line, such as $x = 3$? Experiment and try to find a formula for the x and y coordinates of the image of (x,y). What about a horizontal line?

8. Rotations and Coordinates



1. Write down the coordinates of A, B, and C.

2. Draw the image of the triangle after rotating it around the origin by 90° counterclockwise. Write the coordinates of the vertices.

3. Draw the image of the triangle after rotating it around the origin by 180°. Write the coordinates of the vertices.

4. **Challenge:** If you solved #5 in the previous lesson, you may be ready to answer this question: what happens to the coordinates of a point after a 180° rotation around (3,2)?