Powers, Algebra 1— Teacher Notes

Philosophy

The basic philosophy of these lessons is to *teach for understanding*. Thus:

- The lessons start by describing a situation *without invoking new vocabulary or notation*. This could be a simplified real world context, or just a number pattern. The main point is not so much that they will necessarily discover new concepts, but that they will be better prepared to hear a teacher explanation. If some of them discover the concepts, all the better.
- The particular topics of percent and powers both require *extensive use of the calculator*. This is consistent with the Common Core emphasis on smart use of tools. See the note below on possible exceptions to this.
- Once the concept and any related techniques are introduced, students apply them to other problems.
- Some of the problems involve making a table of change over time. Other problems involve trial and error (guess and check, if you prefer.) These kinds of problems are a form of *practice* that is not boring, or at any rate, less boring than random exercises.
- There is a fair amount of *repetition*, because the ideas are difficult, but hopefully not exact repetition.
- Because some of the work is challenging, allow the students to help each other, and be ready to help them as well, sometimes individually, sometimes through a class discussion.

Once new rules have been learned, post them prominently in the classroom. This way the focus will be on understanding, not memorization. You may at some later date decide to ask for memorization and take them down, but starting with memorization is a recipe for disaster. The kids who can do it feel they can turn off their brain and need not understand what they're doing. This sort of mastery is fragile and does not last. And the kids who have trouble memorizing things they don't understand are frozen out of the lesson altogether. Neither group benefits if you give up on teaching for understanding.

No-Calculator Activities

While the calculator is assumed throughout these lessons, you can and should do some no-calculator activities along the way. For example you can ask the students to figure out mentally or on paper basic power questions, like:

- What is 3^3? 10^3? (you can do a bunch of the easiest ones before you get fancier)
- What is 2^3? What is $2^3 \cdot 2^3$? What is $(2^3)^2$?
- etc.

The purpose of the mental math is two-fold: on the one hand, it shows that one need not run to the calculator every time; on the other hand, it helps kids focus on the meaning of powers in a way that just working with the calculator cannot do.

CCSS-M Alignment

This series of lessons on Exponents is designed to address the following High School standards:

A-SSE:

1 Interpret expressions that represent a quantity in terms of its context.

- a Interpret parts of an expression, such as terms, factors, and coefficients.
- b Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret* $P(1+r)^n$ *as the product of P and a factor not depending on P.*

2 Use the structure of an expression to identify ways to rewrite it. *For example, see* $x^4 - y^4$ *as* $(x^2)^2 - (y^2)^2$*, thus recognizing it as a difference of squares that can be factored as* $(x^2 + y^2)(x^2 - y^2)$.

F-LE:

1 Distinguish between situations that can be modeled with linear functions and with exponential functions.⋆

- a Prove that linear functions grow by equal differences over equal intervals, and that exponential functions grow by equal factors over equal intervals.
- b Recognize situations in which one quantity changes at a constant rate per unit interval relative to another.
- c Recognize situations in which a quantity grows or decays by a constant percent rate per unit interval relative to another.

Because these materials are designed to foster teaching for understanding, they naturally hit on other CCSS standards, especially 8.EE 1. They also prepare students for N-RN 1-2.

Getting Ready

You may choose to precede this work with some or all of the 8th grade Powers packet.

Lesson 1: Working with Monomials

If your students are not familiar with the word "monomial", you will not to introduce it. Give examples and non-examples of monomials, and discuss them, but it is not necessary to spend a huge amount of time on this.

This lesson starts with some rather open-ended questions in order to prepare students for the basic law of exponents: the product of powers law: $x^p \cdot x^q = x^{p+q}$. If this seems too abrupt, and you think your students need a little more time to lead up to this, you might precede this lesson with Lessons 1 and 2 of the 8th grade packet.

Do not announce the rule at the start of the lesson! It is intended to be the final result of working on this sheet. After the students have done the sheet is a good time to make this rule explicit, and post it on the bulletin board.

If there is more time available, it is easy to make up problems of this type. Be sure to include open-ended problems such as #3.

Lesson 2: Power of a Product

This is a short lesson. Work an example "the long way", as in the introduction to the lesson before the students tackle the worksheet. The important part is to model how those steps lead to the result. Only emphasize the rule, and post it on the bulletin board, after the students have worked through the sheet.

You may end with a comparison of the product of powers law and the power of a product law. Which one requires the same base? Which one requires the same exponent?

Lesson 3: Powers and Ratios

Ratios work largely the same way as products, and the lesson is not unlike Lessons 1 and 2. It leads to two different rules: the power of a ratio, and a ratio of powers. The first involves the same exponents; the second involves the same bases.

Some students may get ideas about negative exponents from this worksheet. If so, you might discuss their ideas, and assure them that you will return to this topic in the next lesson.

The lesson ends with some equation solving. These problems cannot be solved by the usual techniques, or at least they should not. The way to get at the value of x is to think about the various laws of exponents, and try to see how they apply to these expressions.

Lesson 4: Negative Exponents, Negative Bases

The meaning of negative exponents is derived from patterns. You can complement this approach by returning to #8-11 of Lesson 3 and seeing which of those problems can yield negative exponents.

Negative bases and parentheses are the source of many mistakes in algebra. A discussion of #7 and #8 might help throw some light on that subject.

Lesson 5: Ratio of Powers

Here the ratio of powers is reviewed, and applied in cases where simplifying ratios yields negative exponents.

Lesson 6: More on Exponential Growth

Again, this lesson may be more accessible if you did Lessons 1 and 2 from the 8th grade packet. Here, we apply negative exponents to make estimates of past values.

Lesson 1: Working With Monomials

In the expression

$$x^3 + 2xy - 3x + 4,$$

four quantities are added or subtracted, so we say that there are four *terms*. x^3, 2xy, 3x, and 4 are the terms. Note that a term is a product of numbers and variables. The number is called a *coefficient*.

The sum or difference of one or more terms is called a *polynomial*.

A polynomial with three terms is called a *trinomial*.
A polynomial with two terms is called a *binomial*.
A polynomial with one term is called a *monomial*.

Note that polynomials do not involve division by variables. For example, (1/x) + x is not a polynomial.

The product of the monomials $3x^2$ and $9x^4$ is also a monomial. This can be shown by using the definition of exponentiation as repeated multiplication:

$$3x^2 = 3 \cdot x \cdot x \text{ and } 9x^4 = 9 \cdot x \cdot x \cdot x \cdot x$$

so

$$3x^2 \cdot 9x^4 = 3 \cdot x \cdot x \cdot 9 \cdot x \cdot x \cdot x \cdot x = 27x^6$$

1. Find another pair of monomials whose product is $27x^6$.

2. **Exploration**. If possible, find at least two answers to each of these problems. Write $27x^6$ as:
   a. the product of three monomials

   b. a monomial raised to a power

   c. the quotient of two monomials

The monomial $48x^9$ can be written as a product in many different ways. For example, $16x^6 \cdot 3x^3$ and $12x^5 \cdot 4x^3 \cdot x$ are both equal to $48x^9$.

3. Write $48x^9$ in three more ways as a product of two or more monomials.

4. Write $35x^4$ as a product in which one of the factors is:
   a. a third degree monomial
   b. a monomial with a coefficient of 7.
   c. $5x^0$
   d. $35x^3$

5. Write x^5 in three ways as a product of two or more monomials.

6. **Generalization**. Study your answers to problem 5. Then, fill in the exponent, and explain:

$$x^{\mathrm{a}} \cdot x^{\mathrm{b}} = x^{?}$$

7. If possible, write each expression more simply. If it is not possible, explain why not.

   a. $3x^5 \cdot 6x^4$

   b. $x^5 \cdot y^7$

   c. $y^7 \cdot y^3$

   d. $4a^4 \cdot 9a^3$

The generalization you made is one of the laws of exponents.
It is sometimes called the *product of powers* law. It says that

$$x^a \cdot x^b = x^{a+b}, \text{ as long as } x \text{ is not } 0.$$

However, notice that it works only when the bases are the same.

Lesson 2: Power of a Product

The expression $x^4 \cdot y^4 \cdot z^4$ is the product of three powers. Since the bases are not the same, we cannot use the product of powers law. However, notice that since the exponents are the same, it is possible to write a product of powers as a single power:

$$\begin{aligned} x^4y^4z^4 &= x \cdot x \cdot x \cdot x \cdot y \cdot y \cdot y \cdot y \cdot z \cdot z \cdot z \cdot z \\ &= xyz \cdot xyz \cdot xyz \cdot xyz \\ &= (xyz)^4 \end{aligned}$$

1. Write $16a^2b^2$ as the square of a monomial. (Hint: First rewrite 16 as a power.)

2. Write p^3q^3 as the cube of a monomial.

3. If possible, write each expression as a single power. If it is not possible, explain why not.

   a. $32n^5m^5$

   b. x^2y^3

   c. $(2n)^7 \cdot (3m)^7$

   d. $(ab)^4 \cdot (bc)^4$

The generalization you used above is another of the laws of exponents.
It is sometimes called the *power of a product* law. It says that

$x^ay^a \cdot = (xy)^a$, as long as x and y are not 0.

However, notice that it works only when the exponents are the same.

4. Write without parentheses.

a. $(6y)^2$

b. $(3xy)^4$

c. $(5xyz)^3$

d. $(2x)^3$

e. $(2xy)^3$

f. $(2xyz)^3$

5. Write $64x^3y^6z^9$ as the cube of a monomial.

Lesson 3: Powers and Ratios

Power of a Ratio

1. Write 49/25 as the square of a ratio.

Study this example:

$$\left(\frac{x}{y}\right)^3 = \frac{x}{y}\cdot\frac{x}{y}\cdot\frac{x}{y} = \frac{x\cdot x\cdot x}{y\cdot y\cdot y} = \frac{x^3}{y^3}$$

This law of exponents is called the *power of a ratio* law. It says that:

$$\frac{x^a}{y^a} = \left(\frac{x}{y}\right)^a$$, as long as x and y are not 0.

However, notice that it works only when the exponents are the same.

2. Write as a power of a ratio:

   a. $8x^3 / y^6$

   b. $16x^4 / x^{10}$

3. Write as a ratio of monomials:

   a. $(5x / 7z)^9$

   b. $(2xy / yz)^2$

Ratios of Monomials

Consider the ratio $6x^5 / 4x^7$, where x is not 0. By multiplying numerator and denominator by x, you can get the equivalent ratio $6x^6 / 4x^8$. Or, you can get an equivalent ratio in lowest terms by noticing that:

$$\frac{6x^5}{4x^7} = \frac{3}{2x^2} \cdot \frac{2x^5}{2x^5} = \frac{3}{2x^2}$$

4. Explain the example above.

5. Write in lowest terms:

   a. $8x^8 / 6x^9$

   b. $7x^7 / 5x^4$

In some cases, a ratio can be simplified to a monomial. For example:

$$\frac{150x^6}{50x^4} = 3x^2$$

6. a. Explain the example.

   b. Write $3x^2$ as a ratio in three other ways.

7. Write $12y^3$ as a quotient of two monomials in which:
   a. one is a 4th degree monomial

   b. one has a coefficient of 5

   c. one is a monomial of degree 0

8. a. Write x^5 as a ratio in three ways.

   b. Find three ratios equivalent to $1/x^5$.

9. **Generalization**. Study your answers to problem 8. Compare the situations in (a) and (b). Explain how to simplify a ratio whose numerator and denominator are powers of x.

10. Fill in the exponent, assuming $p > q$:

$$\frac{x^p}{x^q} = x^?$$

11. Write these ratios in lowest terms:

a. $3x^5 / 6x^4$

b. x^5 / y^7

c. y^3 / y^7

d. $45a^4 / 9a^3$

Solving Equations

Solve for x.

12. a. $\frac{5^{2x}}{5^x} = 5^7$

b. $\frac{(7^2)^x}{7^4} = 7^6$

c. $\dfrac{3^3 \cdot 4^7}{3 \cdot 4^x} = \left(\dfrac{3}{4}\right)^2$

d. $\dfrac{15h^x}{12h^a} = \dfrac{5}{4h^6}$

13. **Challenge:**

a. $\dfrac{(3 \cdot 5)^3}{108 \cdot 5^x} = \dfrac{1}{20}$

b. $\dfrac{3 \cdot 4^{6p}}{9 \cdot 4^x} = \dfrac{1}{3 \cdot 4^{4p}}$

Lesson 4: Negative Exponents, Negative Bases

Reciprocals

In previous lessons, we have considered only whole number exponents. Does a negative exponent have any meaning? To answer this, consider these patterns:

$3^4 = 81$	$(1/3)^4 = 1/81$
$3^3 = 27$	$(1/3)^3 = 1/27$
$3^2 = 9$	$(1/3)^2 = 1/9$
$3^1 = 3$	$(1/3)^1 = 1/3$
$3^0 = ?$	$(1/3)^0 = ?$
$3^{-1} = ?$	$(1/3)^{-1} = ?$

1. a. Look at the powers of 3. How is each number related to the number above it?

   b. Following this pattern, what should the value of 3^{-1} be?

   c. Now look for a pattern in the powers of 1/3. As the exponent increases, does the value of the power increase or decrease?

   d. Following this pattern, what should the value of $(1/3)^{-1}$ be?

   e. Compare the values of 3^{-1}, 3^1, $(1/3)^1$ and $(1/3)^{-1}$. How are they related?

   f. Use the pattern you found to extend the table down to 3^{-4} and $(1/3)^{-4}$.

Another way to figure out the meaning of negative exponents is to use the product of powers law. For example, to figure out the meaning of 3^{-1}, note that:

$$3^{-1} \cdot 3^2 = 3^1$$
$$3^{-1} \cdot 9 = 3$$

But the only number that can be multiplied by 9 to get 3 is 1/3, so 3^{-1} must equal 1/3.

2. Find the value of 3^{-1} by applying the product of powers law to $3^1 \cdot 3^{-1}$.

3. Use the same logic to find the value of
   a. 3^{-2}
   b. 3^{-x}

4. Are the answers you found in problem 3 consistent with the pattern you found in Problem 1? Explain.

5. **Summary**. Many people think that 5^{-2} equals a negative number, such as -25.
   a. Write a convincing argument using the product of powers law to explain why this is not true.
   b. Show how to find the value of 5^{-2} using a pattern like the one in problem 1

6. a. Show that $5x^2$ and $5x^{-2}$ are not reciprocals by showing that their product is not 1.

   b. Find the reciprocal of $5x^2$.

Opposites

The expression $(-5)^3$ has a negative base. This expression means *raise -5 to the third power*.

The expression -5^3 has a positive base. This expression means raise *5 to the third power and take the opposite of the result*.

7. Which of these expressions represent negative numbers? Show the calculations or explain the reasoning leading to your conclusions.

-5^3	$(-5)^3$	-5^2	$(-7)^{15}$	$(-7)^{14}$
-5^{-3}	$(-5)^{-3}$	-5^{-2}	$(-7)^{-15}$	$(-7)^{-14}$

8. a. Is $(-5)^n$ always, sometimes, or never the opposite of 5^n? Explain, using examples.

   b. Is -5^n always, sometimes, or never the opposite of 5^n? Explain, using examples.

Lesson 5: Ratio of Powers

Negative exponents often arise when simplifying ratios of monomials.

This law of exponents is sometimes called the *ratio of powers* law:

$$\frac{x^a}{x^b} = x^{a-b}$$, as long as x is not 0.

However, notice that it works only when the bases are the same.

Examples

$$\frac{x^6}{x^7} = x^{6-7} = x^{-1} \text{ or } \frac{1}{x^1}$$

$$\frac{x^{3a}}{x^{5a}} = x^{3a-5a} = x^{-2a} \text{ or } \frac{1}{x^{2a}}$$

1. Simplify:

   a. $4x^6 / 5x^7$

   b. $2x^8y^3 / 2xy$

   c. y^3 / y^7

   d. $45a / 9a^5$

2. Simplify:

   a. $\dfrac{400a^5}{25a^2}$

   b. $\dfrac{400x^3}{200x^8}$

   c. $\dfrac{3m^6}{9m^3}$

   d. $\dfrac{9R^a}{3R^a}$

3. a. Write as a power of 4: $4^{3+x} / 4^{3-x}$

   b. Write as a power of 7: $7^{5x-5} / 7^{5x-6}$

4. Solve for x.

   a. $\dfrac{7^4}{7^{x+2}} = 7^3$

   b. $\dfrac{3 \cdot 5^{x+2}}{12 \cdot 5^2} = \dfrac{1}{20}$

Lesson 6: More on Exponential Growth

A bacterial culture doubles every hour. At this moment it weighs 10 grams.

1. What will it weigh
   a. in one hour?

   b. in 2 hours?

   c. in 9 hours?

   d. in x hours?

2. What did it weigh
   a. 1 hour ago?

   b. 2 hours ago?

   c. 9 hours ago?

   d. x hours ago?

3. a. Explain why the weight of the bacteria culture x hours from now is given by:

$$W = 10 \cdot 2^x$$

   b. Explain the meaning of substituting a negative value for x.

4. Show your calculations, using the equation in problem 9, to find out:
   a. how much it will weigh in three hours

   b. how much it weighed three hours ago

In 1975, the world population was about 4.01 billion and growing at the rate of 2% per year.

5. If it continued to grow at that rate, write a formula for the world population after x years.

If it had been growing at the same rate before 1975, we could estimate the population in previous years by using negative values of x in the formula.

6. Use your calculator to find the value of $(1.02)^4$ and its reciprocal, $(1.02)^{-4}$.

7. Show your calculations, using the equation in problem 10 to estimate the population in:
   a. 1971

   b. 1979

8. Assume the world population had been growing at this rate since 1925.
   a. Estimate the world population in 1925.

   b. Compare this number with the actual world population in 1925, which was about 2 billion. Was the population growth rate between 1925 and 1975 more or less than 2%? Explain.