Triangle Congruence and Similarity
A Common-Core-Compatible Approach

Henri Picciotto

The Common Core State Standards for Mathematics (CCSSM) include a fundamental change in the geometry program in grades 8 to 10: geometric transformations, not congruence and similarity postulates, are to constitute the logical foundation of geometry at this level. This paper proposes an approach to triangle congruence and similarity that is compatible with this new vision.

Transformational Geometry and the Common Core

From the CCSSM[1]:

> The concepts of congruence, similarity, and symmetry can be understood from the perspective of geometric transformation. Fundamental are the rigid motions: translations, rotations, reflections, and combinations of these, all of which are here assumed to preserve distance and angles (and therefore shapes generally). Reflections and rotations each explain a particular type of symmetry, and the symmetries of an object offer insight into its attributes—as when the reflective symmetry of an isosceles triangle assures that its base angles are congruent.
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> In the approach taken here, two geometric figures are defined to be congruent if there is a sequence of rigid motions that carries one onto the other. This is the principle of superposition. For triangles, congruence means the equality of all corresponding pairs of sides and all corresponding pairs of angles. During the middle grades, through experiences drawing triangles from given conditions, students notice ways to specify enough measures in a triangle to ensure that all triangles drawn with those measures are congruent. Once these triangle congruence criteria (ASA, SAS, and SSS) are established using rigid motions, they can be used to prove theorems about triangles, quadrilaterals, and other geometric figures.
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> Similarity transformations (rigid motions followed by dilations) define similarity in the same way that rigid motions define congruence, thereby formalizing the similarity ideas of "same shape" and "scale factor" developed in the middle grades. These transformations lead to the criterion for triangle similarity that two pairs of corresponding angles are congruent.

This is a substantial departure from the traditional approach, in which congruence and similarity postulates are foundational. I support this change.

Here is a pedagogical argument: congruence postulates are pretty technical and far from self-evident to a beginner. In fact, most teachers introduce the basic idea of congruence by saying something like "if you can superpose two figures, they are congruent." Well, that is not very far from saying "if you can move one figure to land exactly on top of the other, they are congruent." In other words, getting at congruence on the basis of transformations is more intuitive than going in the other direction.

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There are also mathematical arguments for the change:

◊ A transformational approach offers deeper links between algebra and geometry, given the emphasis on functions (and thus composition of functions, inverse functions, fixed points, and so on.)
◊ The natural connections with complex numbers and matrices can immeasurably enhance the teaching of these topics in grades 11-12.[2]
◊ This approach can potentially give symmetry a greater role in school mathematics, which is not only a plus for geometric thinking, but also helps to make connections with art and nature, and provides a context to introduce elements of abstract algebra that throw light on operations. Operations are perhaps the main topic of K-12 mathematics.[3]
◊ Finally, the change makes it possible to discuss the similarity of curves (such as circles and parabolas), which could not be done under the traditional definition of similarity, as it relied on equal angles and proportional sides.[4]

Curricular Implications

One of the consequences of this change is the need for some clarity on how this new vision affects the logical structure of high school geometry. This paper attempts to shed some light on that question.

In my view, the main purpose of teaching geometry in high school is to teach geometry, not to build a formal axiomatic system. The geometry curriculum needs to provide plenty of opportunity for reasoning, without getting stuck in an overly formal system where a rigid format for proof is required. This is especially true at the beginning of the course, where overly formal attempts to establish results that students consider self-evident backfires when it comes to student understanding and motivation. An emphasis on formalism is vastly more effective when students are a couple of years older.

Therefore, I recommend that students have a lot of experience with interesting questions involving geometry (including but not limited to transformations) before being introduced to the ideas in this paper, which is intended mostly for the benefit of teachers and curriculum developers. In particular, one of the implications of the approach outlined herein is that students should do substantial work with geometric *construction* prior to being exposed to these proofs. This work should include traditional tools (compass and straightedge) but also contemporary tools such as patty paper, Plexiglas see-through mirrors, and especially interactive geometry software. Such work is essential in developing students' visual sense and logic, and is a strong preparation for the proofs I present in this paper.[5]

In the bigger picture, I agree with the CCSSM that an informal introduction to the triangle congruence criteria should precede any formal work on this topic, but I fear that seventh grade may be too early to do it effectively. Moreover, I am certain that a semi-formal introduction to similarity should follow, not precede the work on congruence, for both logical and pedagogical reasons.

In any case, I hope this paper is useful even to people who do not share my views on pedagogy.

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[2] See *Computing Transformations* (Henri Picciotto, www.MathEducationPage.org/transformations)
[3] For some introductory curriculum on symmetry, see Chapter 5 of *Geometry Labs* (Henri Picciotto, www.MathEducationPage.org/geometry-labs). For an *Introduction to Abstract Algebra* suitable for grades 8 to 12, see http://www.mathedpage.org/abs-alg/, by Henri Picciotto
[4] See *Geometry of the Parabola* (Henri Picciotto, http://www.mathedpage.org/parabolas/geometry/)
[5] See *Geometric Construction* (Henri Picciotto, http://www.mathedpage.org/constructions/)

Definitions

It is important to not embark on formally defining ideas such as segment length and angle measure at the pre-college level. Such attempts only serve to alienate students, and add nothing to the substance of the course.

In this paper, some definitions are unchanged from a traditional approach to secondary school geometry. For example these two:

The perpendicular bisector of a segment is the perpendicular to the segment through its midpoint.
A circle with center O and radius r is the set of points P such that OP = r.

On the other hand, the following definitions may be new:

Definitions: A *transformation of the plane* is a one-to-one function whose domain and range are the entire plane. An *isometry* is a transformation of the plane that preserves distance.[6]

In this paper, the only isometries we will need are reflections.

Definition: A *reflection* in a line b maps any point on b to itself, and any other point P to a point P' so that b is the perpendicular bisector of PP'.[7]

Definition: A dilation with center O and scaling factor r maps O to itself and any other point P to P' so that O, P, and P' are collinear, and the directed segment OP' = $r \cdot$ OP.[8]

Definition: Two figures are *congruent* if one can be superposed on the other by a sequence of isometries.[9]

Definition: Two figures are *similar* if one can be superposed on the other by a sequence of isometries followed by a dilation.[10]

Note that this definition is equivalent to "Two figures are similar if one can be superposed on the other by a dilation followed by a sequence of isometries." The reason is that if figure 1 is similar to figure 2 by the latter definition, then figure 2 is similar to figure 1 by the former. This observation, and the wording "one can be superposed on the other" make it clear that order does not matter.

[6] In the CCSSM, isometries are called *rigid motions*.
[7] The CCSSM suggests that transformations be defined "in terms of angles, circles, perpendicular lines, parallel lines, and line segments." (p. 76)
[8] In other words, if r is negative, OP' is in the opposite direction from OP. In the CCSSM, no mention is made of the possibility of a negative scaling factor. I believe that it is both mathematically and pedagogically sound to allow for that possibility, especially since that is the assumption of all the interactive geometry software applications.
[9] The concept of superposition goes back to Euclid, and is almost certainly the most common way to introduce the idea of congruence in the classroom. Those who wish to avoid the use of this word in the absence of a formal definition can rephrase the definitions: "Two figures are congruent if one is the image of the other in a sequence of isometries." And likewise for similarity.
[10] I took the definition of congruence almost verbatim from the CCSSM. For similarity, the CCSSM does not specify any particular order for the transformations. I believe that insisting that the dilation be at the end (or beginning) is both easier to understand, and easier to work with in subsequent proofs.

Assumptions

Of course, I will assume the parallel postulate.

I will also make three *construction assumptions*, which I will use without explicitly referencing them:
Two distinct lines meet in at most one point.
A circle and a line meet in at most two points.
Two distinct circles meet in at most two points.

Finally, the following CCSSM-sanctioned assumptions about transformations constitute the foundation of much of this paper:

Assumption 1: Reflection preserves distance and angle measure.

Two immediate consequences of Assumption 1 are[11]:
Congruent segments have equal length.
The corresponding sides and angles of congruent polygons have equal measure.

Assumption 2: If O, A, and B are not collinear, the image A'B' of the segment AB under a dilation with center O and scaling factor r is parallel to AB, with length $r \cdot$ AB.[12]

[11] Another consequence is that rotations and translations preserve distance and angle measure, because they are compositions of two reflections. This can readily proved without recourse to congruent triangles, but I will not include this argument here. I'm only mentioning it because it shows that it is sufficient to make the assumption for reflections. (The CCSSM suggests making the assumption for all rigid motions.)
[12] In *Teaching Geometry According to the Common Core Standards* (Third revision: October 10, 2013) Wu outlines a proof of this result for rational scaling factors. However for the purposes of secondary school math, it is probably best to make this an assumption, as recommended by the CCSSM.

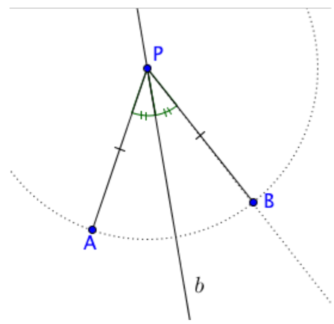
Triangle Congruence

Preliminary Results

Result 0: There is a reflection that maps any given point P into any given point Q.

Proof: If P = Q, reflection in any line through P will do the job. If not, Q is the reflection of P across the perpendicular bisector of PQ.

Result 1: A point P is equidistant from two points A and B if and only if it lies on their perpendicular bisector.



Proof: Given: PA = PB, let us show P must lie on the perpendicular bisector of AB.

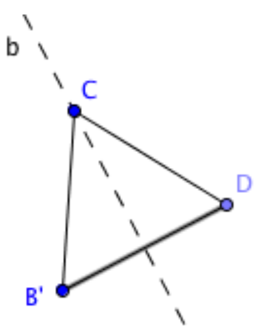
Draw the angle bisector *b* of ∠ APB. If we can show that *b* is the perpendicular bisector of AB, then we are done, since P is on it.

If we were to reflect A in *b*, where would its image A' be? Since reflections preserve angles, A' must be on the ray PB. Since reflections preserve distance, A' must be on the circle centered at P, with radius PA. But the intersection of the ray and the circle is B, so A'=B. It follows that B is the reflection of A in *b*. Therefore *b* is the perpendicular bisector of AB.

Given: P is on the perpendicular bisector *b* of AB. By definition of reflection, A is the image of B, and P is its own image in a reflection across *b,* so PA = PB since reflections preserve distance.

Corollary: If two circles intersect in two points, those points are reflections of each other in the line joining the centers of the circles. (Because the centers are each equidistant from the points of intersection.)

Result 2: If two segments have equal length, then one is the image of the other under either one or two reflections.



Proof: Given AB = CD, by Result 0, we can reflect segment AB so that C is the image of A. Let B' be the image of B. If B'=D, that reflection is the required single reflection. If not, since reflections preserve distance, we have CB'= AB = CD, and by Result 1, C is on the perpendicular bisector *b* of B'D.

Therefore, a reflection of CB' in *b* yields CD. QED.

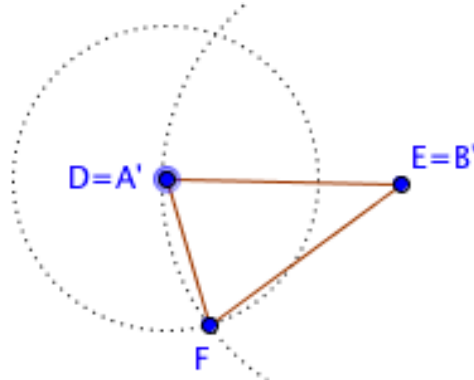
Corollary: Segments are congruent if and only if they have equal length.[13] (This follows from Result 2 and the definition of congruence.)

[13] In general, I have no objection to using "equal" to mean "having equal measure". This usage is reasonably widespread. For example, Chakerian, Stein and Crabill say in their trailblazing text *Geometry: A Guided Inquiry:* "Corresponding parts of congruent triangles are equal." This is also the language used by David E. Joyce in his online version of Euclid's *Elements*. However, it appears that this usage is offensive to many, so I refrain from it in this paper. Still, consistent with Chakerian, Stein, and Crabill, I write AB=CD to mean "the segments AB and CD have equal lengths." If this bothers you, I apologize.

SSS

Theorem: (SSS) If all sides of one triangle have equal lengths, respectively, to all sides of another, then the triangles are congruent.

Proof: We are given ΔABC and ΔDEF, with AB = DE, BC = EF, and AC = DF.

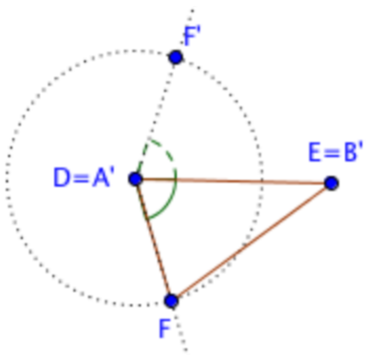


By Result 2, we can superpose AB onto DE in one or two reflections. Because reflections preserve distance, C' (the image of C) must be at the intersection of two circles: one centered at D, with radius DF, the other centered at E, with radius EF.

F, of course, is on both circles. If C' = F, we're done. If not, C' must be at the other intersection, but by Result 1, DE must be the perpendicular bisector of FC', so a reflection across DE superposes the two triangles.

SAS

Theorem: (SAS) If two sides of one triangle have equal lengths to two sides of another, and if the included angles have equal measure, then the triangles are congruent.



Proof: We are given ΔABC and ΔDEF, with AB = DE, AC = DF, and ∠A = ∠D.

By Result 2, we can superpose AB onto DE in one or two reflections. If C'=F, we're done. If not, reflect F across DE.

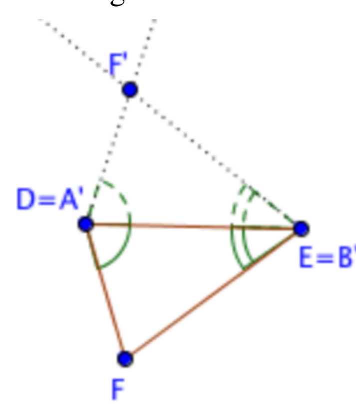
Because reflections preserve distance and angle measure, C' must be on the ray DF' and on the circle centered at D with radius DF. Therefore C' = F', so a reflection across DE superposes the two triangles.

Corollary: Angles are congruent if and only if they have equal measure.

ASA

Theorem: (ASA) If two angles of one triangle are congruent to two angles of another, and if the sides common to these angles in each triangle have equal length, then the triangles are congruent.

Proof: We are given ΔABC and ΔDEF, with AB = DE, ∠A = ∠D, and ∠B = ∠E.



By Result 2, we can superpose AB onto DE in one or two reflections. If C'=F, we're done. If not, reflect F across DE.

Since reflections preserves angle measure, C' must be on the ray DF' and on the ray EF'. It follows that C' = F', so a reflection across DE superposes the two triangles.

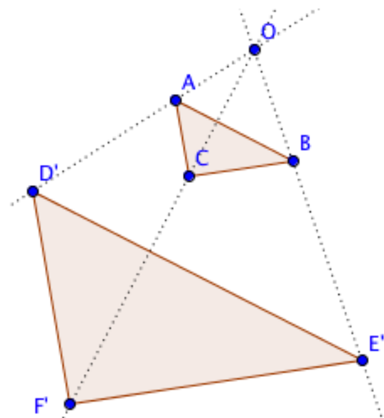
Triangle Similarity

Preliminary Results

Because we have a new definition of similarity, we start by proving that the old definition still applies to similar triangles.

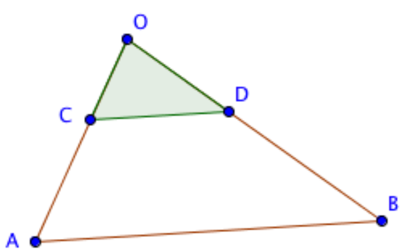
Result 1: Similar triangles have angles with equal measure, and proportional sides.

Proof: Given ΔABC similar to ΔDEF. By definition of similarity, there must be a triangle ΔD'E'F' congruent to ΔDEF, such that it is the image of ΔABC in a dilation. Let us say the dilation has center O and scaling factor *r*.



It follows from Assumption 2 that corresponding sides of ΔABC and Δ D'E'F' are proportional, with ratio *r*. Moreover, we can use what we know about parallels and transversals to show that corresponding angles in those two triangles are equal.[14] Finally, since ΔD'E'F' is congruent to ΔDEF, the same results apply to ΔABC and ΔDEF: their corresponding sides are proportional, and their corresponding angles are equal. QED.

Result 2: If two segments are parallel and unequal, one is the image of the other under a dilation.



Proof: Assume AB // CD and AB ≠ CD. Let O be the intersection of AC and BD, and $r = \dfrac{OC}{OA}$.

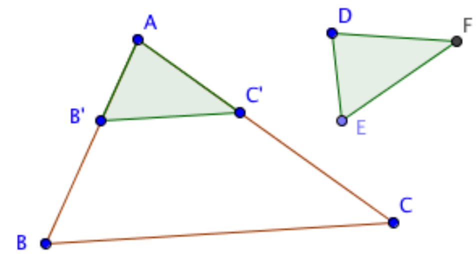
Let A'B' be the image of AB under a dilation with center O and ratio *r*. By construction, A' = C. By Assumption 2, A'B' // AB. Since there is only one parallel to AB through A', it follows that B' = D, and A'B' = CD. Therefore, CD is obtained from AB by a dilation. QED.

[14] In my view, the results about parallels and transversals should be basic assumptions, accepted without a formal proof. In 9th or 10th grade, informal arguments suffice. For example, walking along one of the parallels, turning onto the transversal towards the other parallel, and turning again onto the latter, one's "total turning" is "obviously" either 0°, or 180°. The results follow. If you prefer a formal proof, see Wu (*Teaching Geometry in Grade 8 and High School According to the Common Core Standards*,) but be warned that at the beginning of a course there is no quicker way to confuse and turn off students than to give elaborate arguments in order to arrive at results they consider self-evident.

SSS

Theorem: (SSS similarity) If the sides of two triangles are proportional, then the triangles are similar.

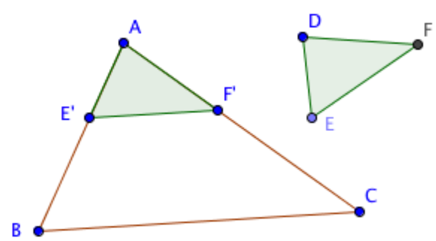
Proof: Assume ΔABC and ΔDEF have proportional sides, with ratio *r*. Dilate ΔABC with center A and ratio *r*.



It follows from the definition of dilation and from Assumption 2 that the sides of the image ΔA'B'C' are equal to the sides of ΔDEF, and so by SSS congruence, these triangles are congruent. Therefore, we can superpose ΔABC onto ΔDEF by a dilation and some isometries. QED.

SAS

Theorem: (SAS similarity) If a pair of sides in one triangle is proportional to a pair of sides in another triangle, and the angles between those sides have equal measure, then the triangles are similar.

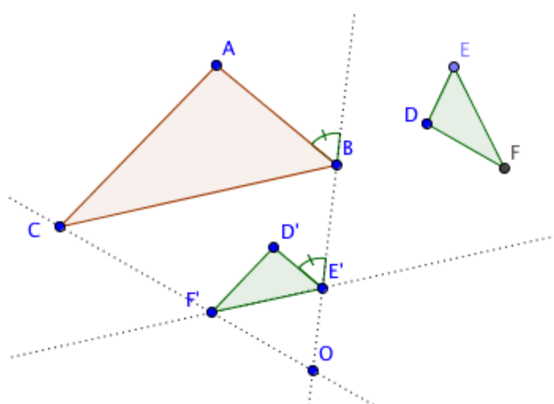


Proof: Given ΔABC and ΔDEF such that $\frac{AB}{DE} = \frac{AC}{DF} = r$ and $\angle A = \angle D$. Assume $r > 1$. (If $r = 1$, we have SAS congruence. If $r < 1$, the argument is nearly identical to the one that follows.) Put point E' on AB so that AE' = DE, and point F' on AC so that AF' = DF.

By definition of dilation, ΔABC is dilated from ΔAE'F' with center A and ratio *r*. But by SAS congruence, ΔAE'F' is congruent to ΔDEF. Therefore, we can superpose ΔABC onto ΔDEF by a dilation and some isometries. QED.

AA

Theorem: (AA) If two angles in one triangle are equal to corresponding angles in another triangle, the triangles are similar.



Proof: Given ΔABC and ΔDEF such that $\angle A = \angle D$ and $\angle B = \angle E$, and therefore $\angle C = \angle F$. Draw a line parallel to BC. Mark two points E' and F' on it, so that E'F' = EF, with E'F' pointing in the same direction as BC. Copy angles E and F to make ΔD'E'F' congruent to ΔDEF by ASA congruence, with D' on the same side of the line as A is to BC. Let O be the intersection of BE' and CF'.

Because $\angle E' = \angle E = \angle B$ in the triangles, and because of equal corresponding angles determined by the transversal OB on the parallels BC and E'F', we conclude that the two angles marked in the figure are equal, and AB // D'E'. Likewise, AC // D'F'.

By Result 2, E'F' is the image of BC under a dilation centered at O, with some scaling factor *r*. Let A' be the image of A under the same dilation. By Assumption 2, A'E' is parallel to AB, and A' must be on D'E'. Likewise, A' must be on D'F'. So A' = D'. Therefore, we can superpose ΔABC onto ΔDEF by a dilation and some isometries. QED.

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