

# **Triangle Congruence and Similarity:**

## **A Common-Core-Compatible Approach**

The Common Core State Standards for Mathematics (CCSSM) include a fundamental change in the geometry curriculum in grades 8 to 10: geometric transformations, not congruence and similarity postulates, are to constitute the logical foundation of geometry at this level. This article proposes an approach to triangle congruence and similarity that is compatible with this new vision.

Here is a pedagogical argument for this change: congruence postulates are rather technical and far from self-evident to a beginner. In fact, many teachers introduce the basic idea of congruence by saying something like "If you can superpose two figures, they are congruent." Well, that is not very far from saying "If you can move one figure to land exactly on top of the other, they are congruent." In other words, basing congruence on transformations is more intuitive than basing transformations on congruence.

There are also mathematical arguments for the change:

- A transformational approach offers deeper links between algebra and geometry, given the emphasis on functions (and thus composition of functions, inverse functions, fixed points, and so on).
- The natural way transformations connect with complex numbers and matrices can immeasurably enhance the teaching of these topics in grades 11-12.
- A transformational approach can give symmetry a greater role in school mathematics, which not only enhances geometric thinking, but also facilitates connections with art and nature.

- Finally, this approach makes it possible to discuss the similarity of curves (such as circles and parabolas), which could not be done under the traditional definition of similarity because it relied on equal angles and proportional sides.

One of the consequences of this change is the need for some clarity on how this new vision affects the logical structure of high school geometry. This article, intended for an audience of mathematics teachers, professional developers, and curriculum writers, attempts to fill that need. While the article is written for adult professionals, we tried to make our approach pedagogically sound —something that might be useful in future curriculum development.

## Definitions

In this article, some definitions are unchanged from a traditional approach to secondary school geometry (for example, the definition of the perpendicular bisector of a segment, or that of a circle). On the other hand, the following definitions may be new:

### Definitions:

- A *transformation of the plane* is a one-to-one function whose domain and range are the entire plane. An *isometry* (or rigid motion) is a transformation of the plane that preserves distance.
- A *reflection* in a line  $b$  maps any point on  $b$  to itself, and any other point  $P$  to a point  $P'$  so that  $b$  is the perpendicular bisector of  $PP'$ .
- A dilation with center  $O$  and scale factor  $k \neq 0$  maps  $O$  to itself and any other point  $P$  to  $P'$  so that  $O$ ,  $P$ , and  $P'$  are collinear and the directed segment  $OP' = k \cdot OP$ . (If  $k$  is negative,  $OP'$  and  $OP$  point in opposite directions. In the CCSSM, no mention is made of the possibility of a negative scale factor, but it is both mathematically and pedagogically sound to allow for that possibility, especially since all the interactive geometry software applications do so.)
- Two figures are *congruent* if one can be superposed on the other by a sequence of isometries.

- Two figures are *similar* if one can be superposed on the other by a sequence of isometries and dilations.

## Assumptions

Of course, we assume the parallel postulate.

We will also make three *construction assumptions*, which we will use without explicitly referencing them:

- Two distinct lines meet in at most one point.
- A circle and a line meet in at most two points.
- Two distinct circles meet in at most two points.

The following two assumptions about transformations constitute the foundation of much of this article. They are consistent with the stronger assumptions suggested in the CCSSM, and in fact can serve as a foundation to prove those.

**Assumption 1:** Reflection preserves distance and angle measure.

An immediate consequence of Assumption 1 is that reflection preserves collinearity. The CCSSM also makes this assumption about translations and rotations, but that is not necessary for our argument here, and moreover, those results can be derived from this assumption.

**Assumption 2:** Dilation preserves collinearity.

The CCSSM assumes the much stronger result that we prove below as the Fundamental Theorem of Dilation.

## Preliminary Results

We start with three preliminary results.

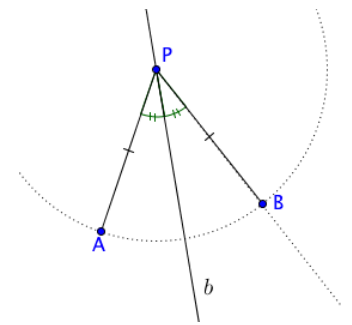
**Theorem 1:** There is a reflection that maps any given point P into any given point Q.

**Proof:** If  $P = Q$ , reflection in any line through  $P$  will do the job. If not,  $Q$  is the reflection of  $P$  across the perpendicular bisector of  $PQ$ .

**Theorem 2:** A point  $P$  is equidistant from two points  $A$  and  $B$  if and only if it lies on their perpendicular bisector.

**Proof:** Given:  $P$  is on the perpendicular bisector  $b$  of  $AB$ . By definition of reflection,  $B$  is the image of  $A$ , and  $P$  is its own image in a reflection across  $b$ , so  $PA = PB$  since reflections preserve distance.

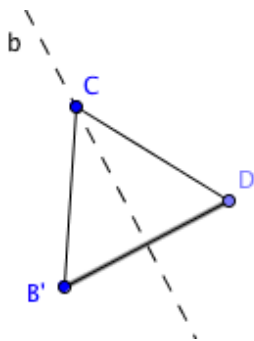
Conversely, given:  $PA = PB$ , let us show  $P$  must lie on the perpendicular bisector of  $AB$ . Draw the angle bisector  $b$  of  $\angle APB$ . If we can show that  $b$  is the perpendicular bisector of  $AB$ , then we are done, since  $P$  is on it.



If we were to reflect  $A$  in  $b$ , where would its image  $A'$  be? Since reflections preserve angles,  $A'$  must be on the ray  $PB$ . Since reflections preserve distance,  $A'$  must be on the circle centered at  $P$ , with radius  $PA$ . But the intersection of the ray and the circle is  $B$ , so  $A'=B$ . It follows that  $B$  is the reflection of  $A$  in  $b$ . Therefore  $b$  is the perpendicular bisector of  $AB$ . QED.

(An indirect proof is also possible if one assumes the triangle inequality.)

**Theorem 3:** If two segments  $AB$  and  $CD$  have equal length, then one is the image of the other with  $C$  the image of  $A$  and  $D$  the image of  $B$ , under either one or two reflections,



**Proof:** Given  $AB = CD$ , by Theorem 1, we can reflect segment  $AB$  so that  $C$  is the image of  $A$ . Let  $B'$  be the image of  $B$ . If  $B' = D$ , that reflection is the required single reflection. If not, since reflections preserve distance, we have  $CB' = AB = CD$ , and by Theorem 2,  $C$  is on the perpendicular bisector  $b$  of  $B'D$ . Therefore, a reflection of  $CB'$  in  $b$  yields  $CD$ . QED.

**Corollary:** Segments are congruent if and only if they have equal length.

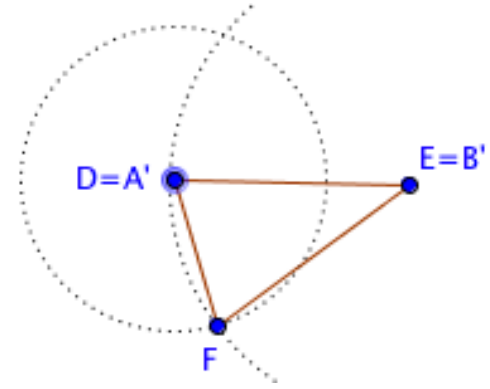
**Proof:** Theorem 3 implies that if segments have equal lengths, they are congruent. The fact that isometries preserve distance guarantees that if segments are congruent, they have equal lengths.

### Triangle Congruence Criteria

**Theorem:** (SSS congruence) If all sides of one triangle are congruent, respectively, to all sides of another, then the triangles are congruent.

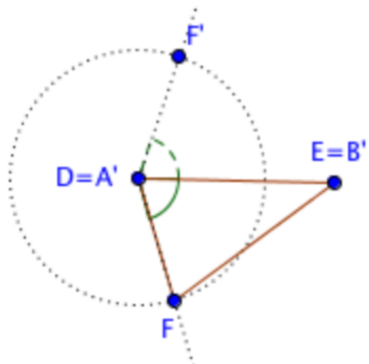
**Proof:** We are given  $\triangle ABC$  and  $\triangle DEF$ , with  $AB = DE$ ,  $BC = EF$ , and  $AC = DF$ .

By Theorem 3, we can superpose  $AB$  onto  $DE$  in one or two reflections. Because reflections preserve distance,  $C'$  (the image of  $C$ ) must be at the intersection of two circles: one centered at  $D$ , with radius  $DF$ , the other centered at  $E$ , with radius  $EF$ .



$F$ , of course, is on both circles. If  $C' = F$ , we're done. If not,  $C'$  must be at the other intersection, but by Theorem 2,  $DE$  must be the perpendicular bisector of  $FC'$ , so a reflection across  $DE$  superposes the two triangles.

**Theorem:** (SAS congruence) If two sides of one triangle are congruent to two sides of another, and if the included angles have equal measure, then the triangles are congruent.



**Proof:** We are given  $\triangle ABC$  and  $\triangle DEF$ , with  $AB = DE$ ,  $AC = DF$ , and  $\angle A = \angle D$ . By Theorem 3, we can superpose  $AB$  onto  $DE$  in one or two reflections, so  $A' = D$  and  $B' = E$ . Because reflections preserve distance,  $C'$  must be on the circle with center  $D$  and radius  $DF$ . Because reflections

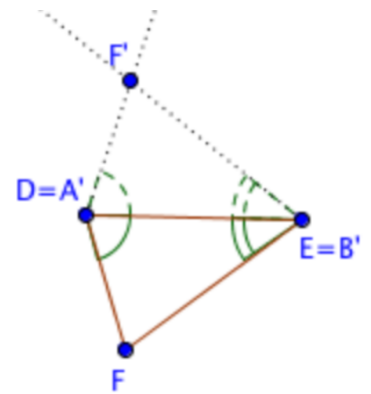
preserve angle measure,  $C'$  must be on one of two rays originating at  $D$ : the one through  $F$ , or its

reflection in DE. In the first case,  $C' = F$  and we're done. Otherwise, to superpose  $\triangle A'B'C'$  onto  $\triangle DEF$ , reflect it across DE. Since reflections preserve distance and angle measure,  $C'' = F$ .

**Theorem:** (ASA congruence) If two angles of one triangle have equal measures to two angles of another, and if the sides common to these angles in each triangle are congruent, then the triangles are congruent.

**Proof:** We are given  $\triangle ABC$  and  $\triangle DEF$ , with  $AB = DE$ ,  $\angle A = \angle D$ , and  $\angle B = \angle E$ .

By Theorem 3, we can superpose AB onto DE in one or two reflections. If  $C' = F$ , we're done. If not, reflect F across DE. Since reflections preserve angle measure,  $C'$  must be on the ray  $DF'$  and on the ray  $EF'$ . It follows that  $C' = F'$ , so a reflection across DE superposes the two triangles.



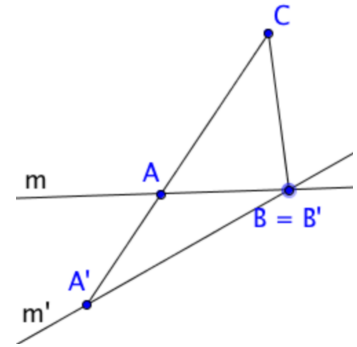
### Fundamental Theorem of Dilations (FTD)

The following result is foundational to the work on dilations and similarity. The CCSSM just requires students to verify this informally, but we offer a proof. The proof uses the fact that opposite sides of a parallelogram are equal, which can of course be established with the help of congruent triangles and parallel lines cut by a transversal.

**Theorem:** (FTD) If C, A, and B are not collinear, the image  $A'B'$  of the segment AB under a dilation with center C and scale factor  $k$  is parallel to AB. If C, A, and B are collinear,  $A'$  and  $B'$  are also on the same line. In either case,  $A'B' = k \cdot AB$ .

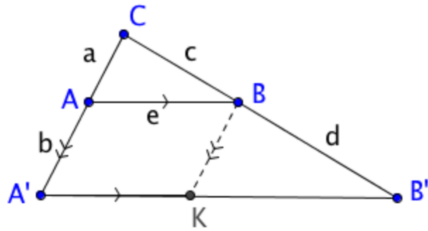
**Proof:** Part 1: If  $m$  passes through C, it follows from the definition of dilation and Assumption 2 that  $m' = m$ .

We now show that under a dilation with center  $C$  and scale factor  $k$ , the image of a line not through  $C$  is parallel to its pre-image. Given a dilation, assume the image of a line  $m$  is  $m'$ . Assume that  $m$  and  $m'$  are not parallel, and meet at a point  $B$ . Choose another point  $A$  on  $m$ . The image of  $A$  is the intersection of line  $CA$  with line  $m'$ .  $B$  is its own image,  $B'$ . From  $A$  and  $A'$ ,  $k \neq 1$ . From  $B$  and  $B'$ ,  $k = 1$ . This contradiction completes the proof: the lines must be parallel.



Part 2: we show that under a dilation with scale factor  $k$ , the length of the image of any

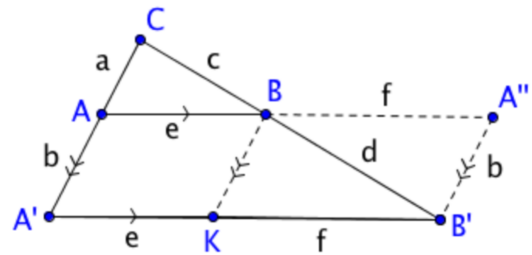
segment is  $k$  times the length of its pre-image. Let  $A'B'$  be the image of  $AB$  in a dilation of center  $C$  (not collinear with  $A$  and  $B$ ), and scale factor  $k$ , with the lengths of various



segments as labeled in the figure. Then  $k = \frac{a+b}{a} = \frac{c+d}{c}$ . By

Part 1,  $AB$  and  $A'B'$  are parallel.

Draw a line through  $B$  parallel to  $AA'$ . Let this line intersect  $A'B'$  at  $K$ . Now dilate  $\triangle ABC$  from center  $B$  with scale factor  $-\frac{d}{c}$ , resulting in this



figure, in which we indicate segment lengths obtained

from the parallelograms. By the definition of dilation,  $\frac{d}{c} = \frac{f}{e}$ . Therefore:  $1 + \frac{d}{c} = 1 + \frac{f}{e}$ , or

$$\frac{c+d}{c} = \frac{e+f}{e}. \text{ Hence } \frac{c+d}{c} = \frac{e+f}{e}. \text{ In terms of the segments, this translates to } \frac{B'C}{BC} = \frac{A'B'}{AB} = k,$$

so  $A'B' = k \cdot AB$ , as was to be proved. (We used the fact that opposite sides of a parallelogram are equal. This can easily be established by using ASA congruence.)

A more straightforward argument based on the distributive rule suffices to prove this result if C, A, and B are collinear.

This follows readily from The FTD allows us to prove that the CCSSM definition of similarity is consistent with the traditional definition:

**Corollary:** Similar triangles have equal angles and proportional sides.

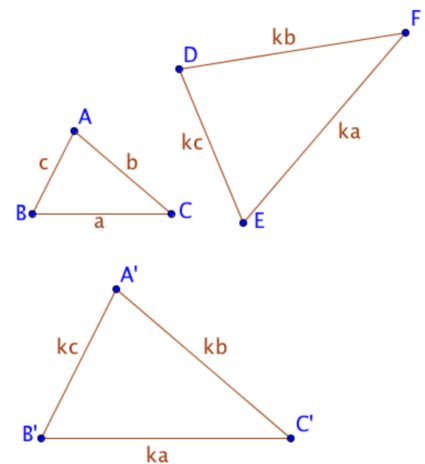
**Proof:** Every isometry preserves side lengths and angle measures. A dilation preserves angle measure because the image of any segment is parallel to or collinear with its pre-image (Part 1 of the FTD). A dilation also preserves ratios of side lengths because the length of every segment is multiplied by the fixed scale factor (Part 2 of the FTD). This completes the proof.

### Triangle Similarity Criteria

**Theorem:** (SSS similarity) If the sides of two triangles are proportional, then the triangles are similar.

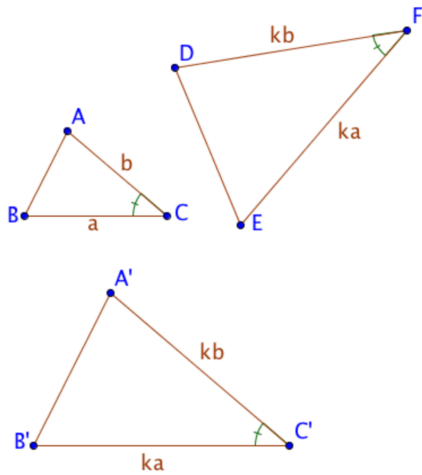
**Proof:** Assume  $\triangle ABC$  and  $\triangle DEF$  have proportional sides, with ratio  $k$ . Dilate  $\triangle ABC$  with any center and scale factor  $k$ , yielding  $\triangle A'B'C'$ .  $\triangle A'B'C'$  is congruent to  $\triangle DEF$  by SSS congruence.

Since we mapped  $\triangle ABC$  to  $\triangle DEF$  by a dilation and a sequence of isometries (those that map  $\triangle A'B'C'$  to  $\triangle DEF$ .) the triangles are similar.





**Theorem:** (SAS similarity) If a pair of sides in one triangle is proportional to a pair of sides in



another triangle, and the angles between those sides are congruent, then the triangles are similar.

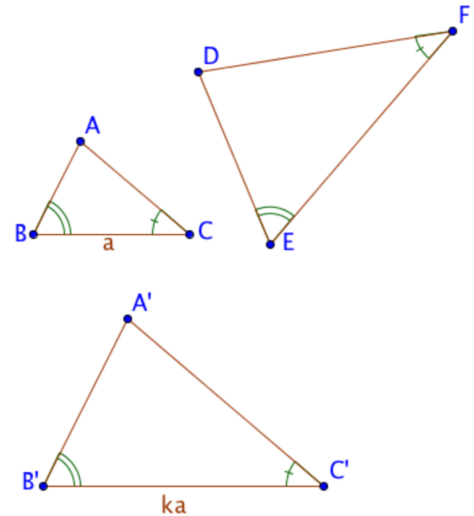
**Proof:** Given  $\triangle ABC$  and  $\triangle DEF$  such that  $\frac{DF}{AC} = \frac{EF}{BC} = k$  and  $\angle C = \angle F$ . Dilate  $\triangle ABC$  with any center and scale factor  $k$ , yielding  $\triangle A'B'C'$ .  $\triangle A'B'C'$  is congruent to  $\triangle DEF$  by SAS congruence. Therefore,  $\triangle ABC$  and  $\triangle DEF$  are similar.

**Theorem:** (AA similarity) If two angles in one triangle are equal to two angles in another triangle, the triangles are similar.

**Proof:** Given  $\triangle ABC$  and  $\triangle DEF$  such that  $\angle C = \angle F$  and

$\angle B = \angle E$ . Say that  $\frac{EF}{BC} = k$  so that  $EF = ka$ . Dilate  $\triangle ABC$

from any point with scale factor  $k$ . The resulting  $\triangle A'B'C'$  is congruent to  $\triangle DEF$  by ASA congruence, and therefore  $\triangle ABC$  is similar to  $\triangle DEF$ .



### Curricular Implications

One of the implications of the approach outlined herein is that students should do substantial work with geometric *construction* prior to being exposed to these ideas. This work should include contemporary tools such as patty paper, Plexiglas see-through mirrors, and especially interactive geometry software, in addition to traditional tools like compass and

straightedge. Work with these tools is powerful in developing students' visual sense and logic, and is a strong preparation for the proofs presented in this article.

In our view, the main purpose of teaching geometry in high school is to teach geometry, not to build a formal axiomatic system. The geometry curriculum needs to provide many opportunities for reasoning without getting stuck prematurely in a too-formal system. This is especially true at the beginning of the course, where excessively formal attempts to establish results that students consider self-evident undermines student understanding and motivation. Therefore, we recommend that students have a lot of experience with interesting questions involving geometry (including, but not limited to, transformations) before being introduced to the ideas in this article.

Given this experience and an understanding of the proofs above, students should be able to apply ideas from the SSS congruence and similarity proofs, for example, to generate their own proofs for the other criteria, perhaps with teacher help. Creating a proof for hypotenuse-leg (HL) congruence and similarity can be done in largely the same way, and can serve as an assessment.

In the bigger picture, we agree with the CCSSM that an informal introduction to the triangle congruence criteria should precede any formal work on this topic, but we fear that seventh grade may be too early to do this introduction effectively. Moreover, we are certain that formal work with similarity should follow formal work on congruence for both logical and pedagogical reasons.

That said, we hope the article will be useful even to teachers and curriculum developers who do not agree with our perspective on pedagogy.