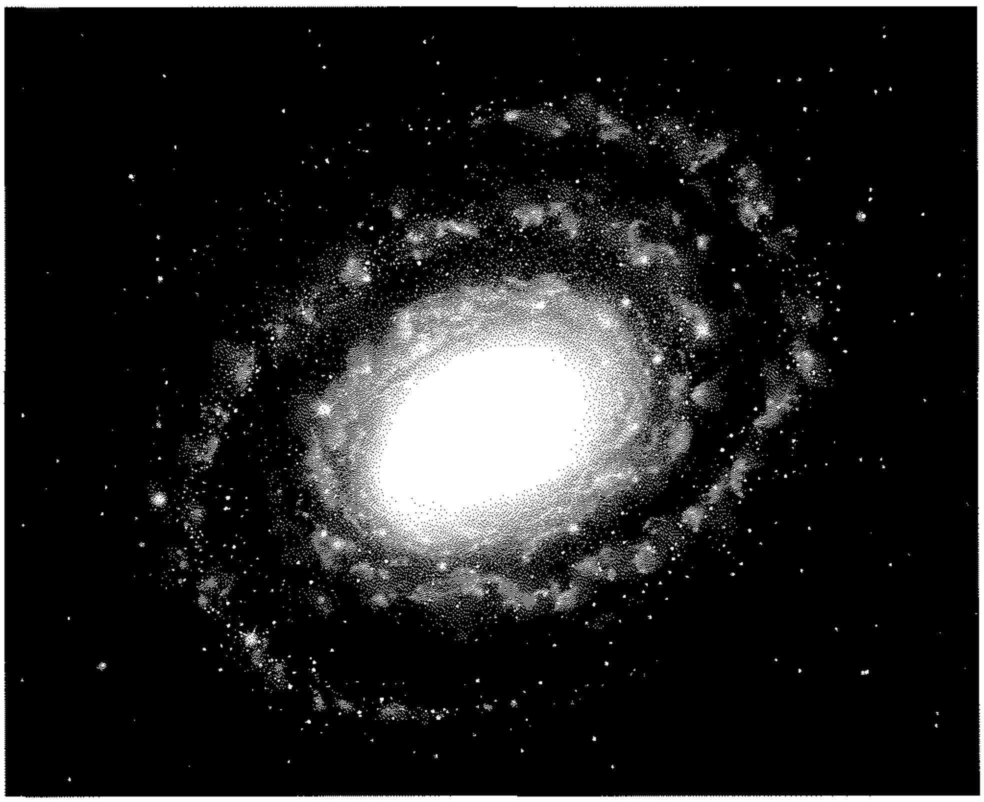
CHAPTER 7



A spiral galaxy, having arms made of gas, dust, and stars

Coming in this chapter:

Exploration The expression $1^3 + 2^3 + 3^3 + 4^3 + 5^3 + \ldots + n^3$ can be modeled by building n cubes out of blocks. Could you rearrange these blocks into a square? If so, what are its dimensions? Experiment with different values of n. Look for a pattern.

Products and Powers

LESSON 7.1

Squares and Cubes

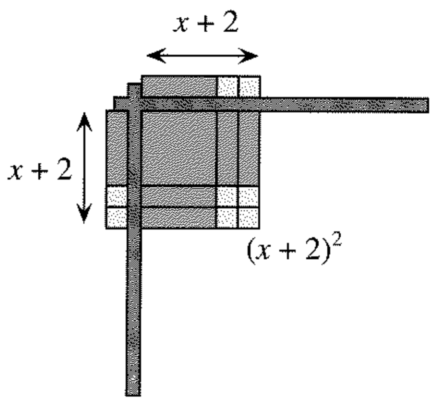
You will need:

the Lab Gear 

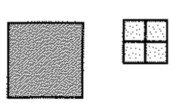
1. **Exploration** Which is greater, $2^2 + 3^2$ or $(2 + 3)^2$? By how much? Which is greater, $5^2 + 8^2$ or $(5 + 8)^2$? By how much? Is it ever true that
$$x^2 + y^2 = (x + y)^2 \text{ ?}$$
How far apart are they? Experiment and write a paragraph summarizing your work and your conclusions. It may help to use the Lab Gear.

HOW MANY SQUARES?

The square $(x + 2)^2$ can be written as the product $(x + 2)(x + 2)$. It can be represented by *a single square* with side $(x + 2)$, as shown in the figure.



The sum of the squares $x^2 + 2^2$ cannot be written as a product or represented with a single square. It must be represented by *two individual squares.*



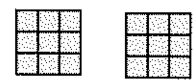
$x^2 + 2^2$

Compare these two expressions.

(i) $2 \cdot 3^2$

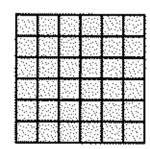
(ii) $(2 \cdot 3)^2$

Because the rules for order of operations tell us to perform exponentiation first, expression (i) means *square 3 and then multiply by 2.* This can be modeled by building two squares with the Lab Gear.



$2 \cdot 3^2$

Expression (ii) means *multiply 2 by 3 and square the result.* Since $2 \cdot 3 = 6$, this can be written more simply as 6^2. This can be modeled by building *one* square with the Lab Gear.



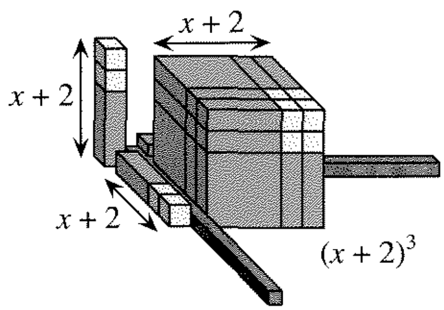
6^2

Make a rough sketch representing each expression, 2-6, with as few squares as possible. Which of these expressions can be modeled as a single square? Which require more than one square? (Be careful!)

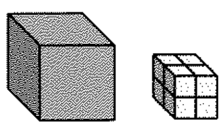
2. a. $(x + 1)^2$ b. $x^2 + 1$
3. a. $4x^2 + 4$ b. $(2x + 2)^2$
4. a. $5^2 + 3 \cdot 5^2$ b. $2^2 + 5 \cdot 2^2$
5. a. $3^2 + 4^2$ b. $(3 + 4)^2$
6. a. $(3 \cdot 4)^2$ b. $3^2 \cdot 4^2$
7. Give the value of each expression.
   a. $3^2 + 4^2$ b. $(3 \cdot 4)^2$
   c. $(3 + 4)^2$ d. $3^2 \cdot 4^2$
   e. $5^2 + 3 \cdot 5^2$ f. $2^2 + 5 \cdot 2^2$

HOW MANY CUBES?

The cube $(x + 2)^3$ can be written as the product $(x + 2)(x + 2)(x + 2)$. It can be represented by *a single cube* with sides $(x + 2)$, as shown.



The sum of the cubes $x^3 + 2^3$ cannot be written as a product. It cannot be represented with a single cube. It must be represented by *two individual cubes.*



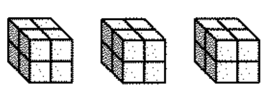
$x^3 + 2^3$

Compare these two expressions.

(i) $3 \cdot 2^3$

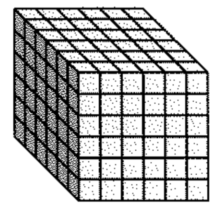
(ii) $(3 \cdot 2)^3$

Because the order of operations tells us to perform exponentiation first, expression (i) means *cube 2 and then multiply by 3.* This can be modeled by building three cubes with the Lab Gear.



$3 \cdot 2^3$

Expression (ii) means *multiply 2 by 3 and cube the result.* Since $3 \cdot 2 = 6$, this can be written more simply as 6^3. This can be modeled by building one cube with the Lab Gear.



6^3

8. What number does each expression equal?
a. $3 \cdot 2^3$ b. $(3 \cdot 2)^3$

How would you represent each expression with as few cubes as possible? It may help to use the Lab Gear. Make a sketch, giving the dimensions of each cube.

9. a. $(x + 1)^3$ b. $x^3 + 1$

10. a. $x^3 + 8$ b. $(x + 2)^3$

11. a. $x^3 + y^3$ b. $(x + y)^3$

Which of these expressions could be modeled using only one cube? Which require more than one cube? Tell how you would represent each expression with as few cubes as possible. Give the dimensions of each cube.

12. a. $6 \cdot 2^3$ b. $(6 \cdot 2)^3$

13. a. $6^3 + 2^3$ b. $(6 + 2)^3$

14. What is the value of each expression?
a. $6 \cdot 2^3$ b. $(6 \cdot 2)^3$
c. $6^3 + 2^3$ d. $(6 + 2)^3$

MAKING SQUARES FROM CUBES

15. a. Use the Lab Gear to show how the expression $1^3 + 2^3 + 3^3$ can be modeled by building three cubes.

b. What was the total number of blocks needed for part (a)?

c. Make a square by rearranging the blocks you used to make the three cubes. What are the dimensions of the square?

16. a. The expression $1^3 + 2^3 + 3^3 + 4^3$ could be modeled by building four cubes. What is the total number of blocks used?

b. How would one make a square by rearranging these blocks? Give the square's dimensions.

17. Compare your answers to problems 15 and 16. Look for a pattern. Check it for $1^3 + 2^3$. Predict the value of the sum,

$$1^3 + 2^3 + 3^3 + 4^3 + 5^3.$$

Check your prediction.

18. **Generalization** The expression

$$1^3 + 2^3 + 3^3 + 4^3 + 5^3 + ... + n^3$$

can be modeled by building n cubes out of blocks. Could you rearrange these blocks into a square? If so, what would its dimensions be? Explain your answer.

REVIEW *CUBING WITH A TABLE*

To find the cube of a polynomial, first find its square, then multiply the result by the polynomial. For example, to calculate $(x + 2y)^3$, first square $x + 2y$.

	x	$2y$
x	x^2	$2xy$
$2y$	$2xy$	$4y^2$

Combine like terms in the body of the table. Multiply this result by $x + 2y$.

	x^2	$4xy$	$4y^2$
x	x^3	$4x^2y$	$4xy^2$
$2y$	$2x^2y$	$8xy^2$	$8y^3$

So $(x + 2y)^3 = x^3 + 6x^2y + 12xy^2 + 8y^3$.

19. Find the cube.

a. $(x + 1)^3$ b. $(2x + 2)^3$

c. $(x + y)^3$ d. $(2x - y)^3$

e. $(3x + 2y - 5)^3$

LESSON 7.2

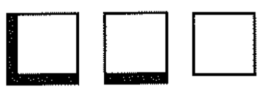
Square Windows

You will need:

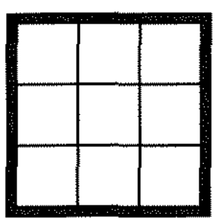
graph paper

THREE TYPES OF PANES

The A.B. Glare window store has started selling a new kind of window. These windows can be made to order by combining three types of square window panes. Each pane measures one foot on each side. The three types of panes are shown below: corner panes, edge panes, and inside panes.



A 3-foot-by-3-foot window is shown below. It was made by putting together 4 corner panes, 4 edge panes, and 1 inside pane.



1. Sketch a 4-foot-by-5-foot rectangular window. How many panes of each type were used to make it?

SQUARE WINDOWS

2. **Exploration** An architect was asked to design a recreation hall. He was going to use the A.B. Glare window panes described above. The building code imposes a limit of 72 square feet for the total area of all windows in the main part of the hall. The architect decides to consider various combinations of square windows such that their total area is exactly 72. Find several such configurations, and for each one, find the total number of each type of pane the architect will need.

The architect is not the only one to like square windows. To save time when customers ask for them, Lara is assembling kits with the correct number of corner panes, edge panes, and inside panes to make square windows of various sizes.

3. Make a table to show how many panes of each type are needed for a 2-by-2 window, a 3-by-3 window, and so on, up to a 10-by-10 window.

4. Study the table from problem 3. Which increases the fastest: the number of corner, edge, or inside panes? Which increases the most slowly? Why?

5. Make three graphs of the data in your table, on the same set of axes.
   a. Graph the number of corner panes as a function of the length of the side of the window. For example, since a 3-by-3 window uses four corner panes, the point (3, 4) would be on your graph.
   b. Graph the number of edge panes as a function of the side length.
   c. Graph the number of inside panes as a function of the side length.

6. Study your graphs. Which is the steepest? Explain why.

7. **Generalization**

a. Write a formula for the number of panes of each type in an x-by-x window. Explain each formula in reference to a sketch of such a window.

b. How are the formulas related to the graphs in problem 5?

8. Add up the algebraic expressions for the numbers of each type of pane. If you did your work correctly, the sum should be very simple.

9. Find the number of corner, edge, and inside panes needed for a 100-by-100 window.

COMPARING SIZES

10. Lara has too many window kits of some types and not enough of other types. She has too many kits for 2-by-2 windows and not enough for 3-by-3 windows. How many panes of each type would she have to add to a 2-by-2 kit to convert it to a 3-by-3 kit?

11. Answer question 10 if Lara wanted to convert

a. a 5-by-5 kit to a 6-by-6 kit;

b. an 8-by-8 kit to a 9-by-9 kit.

12. **Generalization** How many panes of each type would Lara have to add if she wanted to convert an N-by-N-foot kit to an $N+1$-by-$N+1$-foot kit. Explain, using a sketch of an $N+1$-by-$N+1$ window.

13. How many panes of each type would Lara have to add if she wanted to convert an N-by-N-foot kit to an $N+M$-by-$N+M$-foot kit. Explain, using a sketch of an $N+M$-by-$N+M$ window.

MAKING THE MOST OF INVENTORY

14. Suppose you have 12 panes of each type in inventory.

a. What is the largest square window you could make? Give the size of the window and tell how many panes of each type you would have left over.

b. What is the largest square window you could make with the remaining panes? Continue until no more windows can be made. Give the size of all the windows and the number of each type of pane left at the end.

15. Repeat problem 14 for:

a. 20 panes of each type;

b. 100 panes of each type.

16. Now assume that instead of trying for the largest possible square window, you try to make any number of square windows, with the goal of having *as few panes as possible left over.*

a. If you start with 100 panes of each type, what size windows should you make? What will be left over?

b. Compare your answers with other students' answers.

PREVIEW *BIGGER WINDOWS*

17. Suppose each pane, regardless of the type, costs $1.00.

a. Make a table and a graph showing the cost of the window as a function of the side length.

b. Al knows that an 8-by-8 window costs $64.00. He thinks that a 16-by-16 window should cost twice as much, but he isn't sure. What do you think? Explain your opinion.

c. A 16-by-16 window costs how many times as much as an 8-by-8 window?

LESSON 7.3

Squares of Sums

You will need:

the Lab Gear 

1. Exploration

a. Model the square $(x + 1)^2$ with the Lab Gear. Then add blocks to create the square $(x + 2)^2$. What blocks did you need to add to the first square to get the second? Now add blocks to create the square $(x + 3)^2$. What blocks did you add this time? Continue to make the square grow, keeping an organized record of what blocks you add each time. Write a paragraph about any patterns you notice.

b. If a and b are whole numbers, what blocks would you need to add to $(x + a)^2$ to get $(x + a + 1)^2$? To get $(x + a + b)^2$?

MISSING TERMS

2. a. Use the Lab Gear to build a square using 10 x-blocks and any other blocks that you want (except more x-blocks). Sketch the square.

b. What is the area of the square?

c. What are its dimensions?

d. Is this the only such square you could build? (That is, is your answer *unique*?) If it isn't, try to find another possibility. If you can't build another square, explain why.

3. Repeat problem 2, using 16 one-blocks and any other blocks that you want (except more yellow blocks).

4. Repeat problem 2, using 8 xy-blocks and any other blocks that you want (except more xy-blocks).

5. Can you build a square starting with 3 x^2-blocks, if you can use any other blocks except more x^2-blocks? Explain.

6. Can you build a square starting with 15 one-blocks, if you can use any other blocks except more one-blocks? Explain.

7. Build two different squares starting with 4 x^2-blocks, using any other blocks except more x^2-blocks. Are there more solutions? Explain.

TERMS AND COEFFICIENTS

8. a. Use the Lab Gear to build three squares of the form $(x + b)^2$, using a different value of b each time. Sketch the squares.

b. Write the area of the square next to each sketch, combining like terms.

c. Notice how many terms are in each expression for area. Notice the coefficient of each term. Describe what you notice.

In each expression below, a binomial is squared. Distribute and combine like terms.

9. $(2y + 3)^2$ **10.** $(3x + 2)^2$

11. $(2x + 3y)^2$ **12.** $(3x + 2y)^2$

13. Refer to problems 9-12 to answer these questions.

a. How many terms are in each product, after combining like terms?

b. For each binomial, notice the coefficients of each of the terms. Then notice the coefficients in the related expression for area. Describe any relationships you notice.

c. For each binomial, notice the degree of each of the terms. Then notice the degree of each term in the related expression for area. Describe any relationships you notice.

14. Summary Summarize the patterns for the square of a binomial.

15. Generalization The patterns you found can be generalized by using letters instead of numbers for coefficients. Show how you would find the area of a square having side

a. $a + b$; b. $ax + b$;
c. $a + by$; d. $ax + by$.

16. In each expression below, a binomial is squared. Distribute and combine like terms.

a. $(m + n)^2$ b. $(11m + 2)^2$
c. $(5y + 6x)^2$ d. $(1 + 9y)^2$

RECOGNIZING PERFECT SQUARES

$x^2 + 14x + 49$ is called a *perfect square trinomial.* It is the square of the binomial $(x + 7)$, as you can see by writing it in a multiplication table.

	x	7
x	x^2	$7x$
7	$7x$	49

17. Which of the following are perfect square trinomials? For each one, write the binomial it is the square of.

a. $x^2 + 16x + 16$
b. $x^2 + 4x + 4$
c. $x^2 + 10x + 25$
d. $x^2 + 10xy + 25y^2$

18. All of these are perfect square trinomials. Write each one as the square of a binomial. Sketches may help.

a. $4x^2 + 20xy + 25y^2$
b. $36y^2 + 12xy + x^2$
c. $y^2 + 18y + 81$
d. $25x^2 + 10xy + y^2$

19. None of these expressions is a perfect square trinomial. In each one, change just one of the terms to convert the whole expression into the square of a binomial.

a. $4x^2 + 12x + 10$
b. $2x^2 + 8x + 16$
c. $36x^2 + 30x + 25$
d. $1.44x^2 + 1.6x + 2.25$

20. Summary Explain how to recognize a perfect square trinomial. You may use sketches, but be sure to discuss *coefficients, terms,* and *degree.*

21. Look at each perfect square trinomial in this lesson. For each one, find the sum of the coefficients. What do you notice? Explain.

PREVIEW ***HOW MANY TERMS?***

22. **Exploration** Two of the following problems are impossible. Solve the other three. Find a pair of binomials such that their product has:

a. three terms

b. four terms

c. five terms

d. one term

e. two terms

REVIEW ***LAB GEAR MULTIPLICATION***

For each of these problems, 23-25:

a. Use the corner piece to show the multiplication.

b. Check that the resulting figure includes an *uncovered rectangle* of the required dimensions.

c. Write a *length times width equals area* equation.

23. $(y + 2)(y + 2)$ **24.** $(y + 2)(y - 2)$

25. $(y - 2)(y - 2)$

26. Which of the uncovered rectangles in problems 23, 24, and 25 are squares?

DISCOVERY ***CONSTRAINED NUMBERS***

27. What are m and n if they are whole numbers and

a. $89 = 12m + n$, with $n < 12$;

b. $123 = 45m + n$, with $n < 45$;

c. $2345 = 67m + n$, with $n < 67$.

28. If N and m are whole numbers, and $N = 7m + n$, find several values of N such that $n = 2$.

LESSON 7.4

Differences of Squares

You will need:
- the Lab Gear
- graph paper
- scissors

1. **Exploration** Which is greater, $3^2 - 2^2$ or $(3 - 2)^2$? Which is greater, $8^2 - 5^2$ or $(8 - 5)^2$? Is it ever true that

$$y^2 - x^2 = (y - x)^2 \text{ ?}$$

Is it ever true that

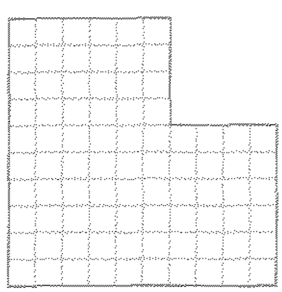
$$y^2 - x^2 < (y - x)^2 \text{ ?}$$

Experiment, and write a paragraph summarizing your work and your conclusions.

CUTTING A SQUARE OUT OF A SQUARE

Problems 2-4 show how to model the difference of two squares geometrically.

2. Cut a 10-by-10 square out of graph paper. Then, out of the corner of this square, cut a 4-by-4 square. The remaining paper should look like this.



3. The size of the remaining paper represents the *difference* of the 10-by-10 square and the 4-by-4 square. Its area is $10^2 - 4^2$ square centimeters. How many square centimeters is this?

4. The odd-shaped figure you have left after cutting out the 4-by-4 square can be rearranged into a rectangle. You can do this by making a single cut in the paper. Try it. Sketch the resulting rectangle and label its length and width.

5. **Generalization** Repeat problems 2-4 for some other differences of squares. (For example, try cutting a 3-by-3 square out of a 7-by-7 square. Try several others.) Can the resulting shape always be rearranged into a rectangle, no matter what two numbers you use? Can you use fractions? What are the dimensions of the rectangle? If it can always be arranged into a rectangle, explain why. If not, explain when it is possible and when it is not possible. Give examples, using sketches.

USING VARIABLES

Use the Lab Gear to do problems 6-9.

6. Trace the x^2-block on a piece of paper and cut out the square. Then trace a 1-by-1 square in the corner of the x^2-paper and cut it out. What difference is represented by the remaining paper?

7. Show how you can rearrange the remaining paper into a rectangle. Make a sketch showing the dimensions of the rectangle.

8. Repeat problems 6 and 7 for the following squares. You do not have to do the actual cutting unless you want to, but your sketches should be traced in the correct sizes.

a. Cut a square having area 4 out of a square having area x^2.

b. Cut a square having area x^2 out of a square having area y^2.

c. Cut a square having area 9 out of a square having area y^2.

d. Cut a square having area 25 out of a y-by-y square.

9. **Generalization** Make a sketch showing what remains after a square having area a^2 has been cut out of a square having area b^2. Then show by sketching how this can be rearranged into a rectangle. What are the dimensions of this rectangle?

FACTORING A DIFFERENCE OF SQUARES

When you cut a square out of a square, the area of the remaining paper is the *difference* of the two squares. When you rearrange this paper into a rectangle and write the area as *length* · *width,* you are writing this difference as a *product,* or factoring. Later in this course you will find this factoring technique helpful in solving equations.

10. Which of these is a difference of two squares?

a. $4x^2 - 16y^2$

b. $4x^2 + 16y^2$

c. $(x - y)(x - y)$

d. $(a - b)^2$

11. Write these differences as the product of two factors.

a. $x^2 - 9$ b. $y^2 - 25$

c. $25 - x^2$ d. $4x^2 - 16$

12. Factor.

a. $9y^2 - 25$ b. $9 - 25x^2$

c. $9y^2 - 25x^2$

13. **Generalization** In this lesson you found a technique for factoring a difference of two squares. However, in all the examples you have done, you have assumed that the first square was larger than the second. Does the pattern work if the first square is smaller than the second? That is, if a is less than b, is it still true that

$$a^2 - b^2 = (a - b)(a + b)?$$

Experiment, using some numbers, and explain your conclusions.

REVIEW THE LAB GEAR MODEL

14. Use the corner piece to multiply $(y + 5)(y - 5)$. Remember to simplify.

15. Show $y^2 - 25$ with the Lab Gear. Show how you can add zero and rearrange the blocks so that the uncovered part forms a rectangle. What are the dimensions of the rectangle?

16. Explain how one can use the Lab Gear to factor

a. $x^2 - 1$; b. $y^2 - x^2$.

17. **Summary** Write a paragraph summarizing what you learned in this lesson about differences of squares. Use sketches and examples.

18. Arrange Lab Gear blocks to show a square having area $(x + 5)^2$.

a. Using the blocks, remove a square having area x^2 out of the square having area $(x + 5)^2$, and rearrange the remaining blocks as a rectangle. Write its dimensions.

b. Repeat part (a) and remove a square having area 25.

c. What other squares can you remove from $(x + 5)^2$? Remove one, and rearrange the remaining blocks into a rectangle.

d. Explain how parts (a), (b), and (c) are examples of the pattern you learned about earlier in this lesson.

19. Write each difference as a product of two factors.

a. $(y + 4)^2 - y^2$

b. $(y + 4)^2 - (y + 3)^2$

c. $(y + 4)^2 - (y + 1)^2$

20. Factor. $(y + 2)^2 - (x + 5)^2$

REVIEW *SOLVING EQUATIONS*

Solve these equations using the cover-up method.

21. $\frac{5 - x}{7} = \frac{8}{14}$

22. $2 - \frac{x - 2}{3} = \frac{2}{3}$

23. $3 + \frac{2 + x}{5} = \frac{19}{5}$

24. $\frac{-7}{6} = \frac{x}{4}$

25. $6 - \frac{14}{x} = \frac{5}{2}$

26. $\frac{2 + x}{8} = \frac{5}{3}$

27. $\frac{1}{x} = 2$

28. $\frac{1 + x}{3} = \frac{2}{9}$

29. $\frac{4}{x} = 5$

30. $\frac{4}{x - 1} = 5$

31. $\frac{4}{3x - 1} = 5$

32. $\frac{4}{x + 4} = \frac{5}{6}$

THINKING WRITING

7.A Cube Problems

THE PAINTED CUBE

Lea made a cube by gluing together 27 Lab Gear 1-blocks.

1. Make a sketch of what this cube would look like. What are its dimensions?

Lea painted the cube red on all six sides. Later, Mary and Martin were annoyed when they discovered what Lea had done. They needed the 27 one-blocks to do a hard factoring problem. Besides, they didn't think she should have been gluing and painting Lab Gear blocks.

2. a. When Mary and Martin broke Lea's cube apart into the 27 original small cubes, how many of the 1-blocks did they find to be painted red on three sides?

b. How many were painted red on only one side?

c. How many were painted red on two sides?

d. How many had no red paint on them?

3. Repeat problem 2 for a 4-by-4-by-4 cube.

4. **Report** Write a report about problems 2 and 3. It should include, but need not be limited to, the following:

- Show how you solved problems 2 and 3. Include sketches.
- Look for patterns in your answers. Use them to guess the answers for a cube of side 5 and a cube of side 6. How can you check whether or not you are right?
- Make a generalization to an n-by-n-by-n cube. Write expressions in terms of n for the number of cubes with 0 sides painted, 1 side painted, 2 sides painted, and 3 sides painted. (Explain why the four expressions should add up to n^3, and check that they do.)

CUBES IN CUBES

It is easy to see that there are 27 different 1-by-1-by-1 cubes in this 3-by-3-by-3 cube. It is harder to see how many different 2-by-2-by-2 cubes there are, because they overlap.

5. Figure out how many different 2-by-2-by-2 cubes there are in a 3-by-3-by-3 cube.

6. Think about a 4-by-4-by-4 cube. It contains how many

a. 1-by-1-by-1 cubes?

b. 2-by-2-by-2 cubes?

c. 3-by-3-by-3 cubes?

d. 4-by-4-by-4 cubes?

e. cubes altogether?

7. Find how many cubes of each size there are in a 5-by-5-by-5 cube. Try to figure out a systematic way for counting the cubes.

8. **Report** Write a report about these cube problems. It should include, but not be limited to, the following:

- Describe the strategy you used to answer problems 5, 6, and 7. Use sketches and explain your reasoning.
- Make a generalization. In an n-by-n-by-n cube, how many cubes of each size (1-by-1-by-1, 2-by-2-by-2, 3-by-3-by-3, and so on) would there be? Write expressions in terms of n.
- Test your generalization by trying it for a 7-by-7-by-7 cube. How many smaller cubes of each size should there be, according to your generalization? If you add all these numbers, do you get the correct total of 784? Show your work.

LESSON 7.5

Remarkable Identities

You will need:
- the Lab Gear
- graph paper
- graphing calculator (optional)

REVIEW *MULTIPLYING PATTERNS*

1. Find these products.
a. $(y - 3)(y + 3)$
b. $(y + 5)(y - 5)$

2. What is the pattern in problem 1?

3. Does the pattern still hold for $(2x - 1)(2x + 1)$? Explain.

4. Find these squares.
a. $(y - 3)^2$ b. $(y - 5)^2$
c. $(y + 3)^2$ d. $(y + 5)^2$

5. What is the pattern in problem 4?

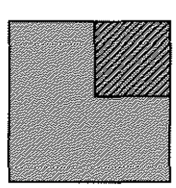
6. Does the pattern still hold for $(2x - 1)^2$ and $(2x + 1)^2$? Explain.

THREE IDENTITIES

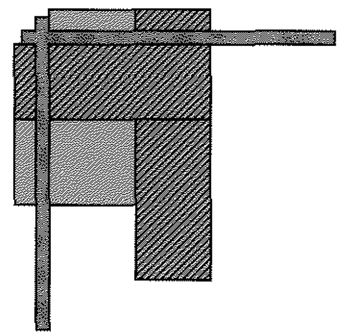
7. True or False? The square of a sum is equal to the sum of the squares. Explain, using a sketch.

8. Describe a shortcut for finding the square of a sum.

The expression $y^2 - x^2$ is the difference of squares. (Remember that shaded blocks are *upstairs.*)



The expression $(y - x)^2$ is the square of a difference.



9. True or False? The square of a difference is equal to the difference of the squares. Explain.

10. Describe a shortcut for finding the square of a difference.

11. Find the products.
a. $(y + x)^2$
b. $(y - x)^2$
c. $(y - x)(y + x)$

As you know, identities are algebraic statements that are always true. The three that are shown in problem 11 are especially important and useful. You should memorize them. For example, using the second one,

$$\begin{aligned}(2x - 5)^2 &= (2x)^2 - 2(2x)(5) + 5^2 \\ &= 4x^2 - 20x + 25.\end{aligned}$$

12. Multiply by using one of the identities. You may check your answers with the Lab Gear or by setting up the multiplication as a table.
a. $(3x - 2)^2$
b. $(3x + 2)^2$
c. $(3x - 2)(3x + 2)$

Even if you don't use the identities for multiplying, it is useful to memorize them in order to recognize them quickly when trying to factor a trinomial. Knowing them is also useful for understanding the solution of quadratic equations.

A CASE OF MISTAKEN IDENTITY

Some of the most common mistakes of math students concern the identities you have learned about in this chapter. Even after having learned the identities, students often forget and write $(x + y)^2 = x^2 + y^2$ or $(x - y)^2 = x^2 - y^2$. This mistake causes math teachers to tear their hair in desperation.

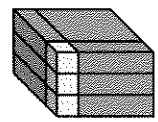
13. **Report** Write an article or create a poster that you think would help other students avoid these mistakes. (Math teachers all over the world would greatly appreciate your help.) Include explanations, sketches, and examples. Make your article or poster appealing, eye-catching, or humorous so that other students will want to read it.

FACTORING

14. Factor these trinomials.
- a. $9x^2 + 6x + 1$
- b. $x^2 - 6xy + 9y^2$
- c. $4x^2 + 4xy + y^2$
- d. $9x^2 - 25$
- e. $4x^2 - 4y^2$
- f. $a^2x^2 + 2acx + c^2$

15. Use the Lab Gear to make as many different rectangles as you can with $3x^2 + 12x + 12$. Write a product corresponding to each rectangle.

16. The figure below shows a box with a square base.
- a. Write an expression for the volume of the box in the form *Height* · *Area of Base.*
- b. Write an expression for the volume of the box in the form *Height* · $(Side)^2$.



17. Each of these expressions gives the volume of a box that has a square base. For each one, write an expression of the form *Height* · $(Side)^2$. You may want to use the Lab Gear.
- a. $3x^2 + 12x + 12$
- b. $8x^2 + 8x + 2$
- c. $3x^2 + 6xy + 3y^2$
- d. $2y^2 + 12y + 18$
- e. $xy^2 + 2xy + x$

18. Each of these polynomials gives the volume of a box that has a square base. For each one, write an expression of the form *Height* · $(Side)^2$, without using the blocks. (Hint: The height of the blocks is the factor that is common to all three terms.)
- a. $27x^2 + 54x + 27$
- b. $60y^2 + 60y + 15$
- c. $50x^2 + 100xy + 50y^2$
- d. $16y^2 + 96y + 144$
- e. $6x^2y + 24xy + 24y$

SQUARING TRINOMIALS

Do you think there is a pattern for the square of trinomials? Experiment with these problems.

19. $(x + y + 2)^2$

20. $(x + y - 5)^2$

21. Describe the pattern you discovered in problems 19 and 20.

22. What is $(a - b + c)^2$ equal to? Use the pattern you discovered, then check your answer by using the distributive law very carefully.

CUBES OF SUMS

23. Find an identity for the cube of a sum. Lab Gear models using 3-D blocks may help. Explain why the cube of a sum is not the sum of the cubes.

PUZZLE *SUM OF SQUARES*

24. $5x^2 + 20x + 25$

Think of the Lab Gear blocks representing this polynomial. The polynomial is not a perfect square, so you cannot rearrange it into a single square. However, it can be arranged into a *sum of squares*. Figure out how you would do it.

REVIEW/PREVIEW *ALWAYS, SOMETIMES, OR NEVER TRUE?*

25. On the same axes, graph $y = 12 - x$ and $y = 8 - x$.

26. Always, sometimes, or never true? (Explain your reasoning in each case.)

a. $12 - x > 8 - x$

b. $12 - x > 13$

c. $8 - x > 12 - x$

d. $4 > 8 - x$

e. $-4 > 8 - x$

27. Always, sometimes, or never true? (If sometimes true, give the values of x that make it true.)

a. $x > 2x - 8$

b. $2x - 5 > 2x - 8$

c. $x < 2x - 5$

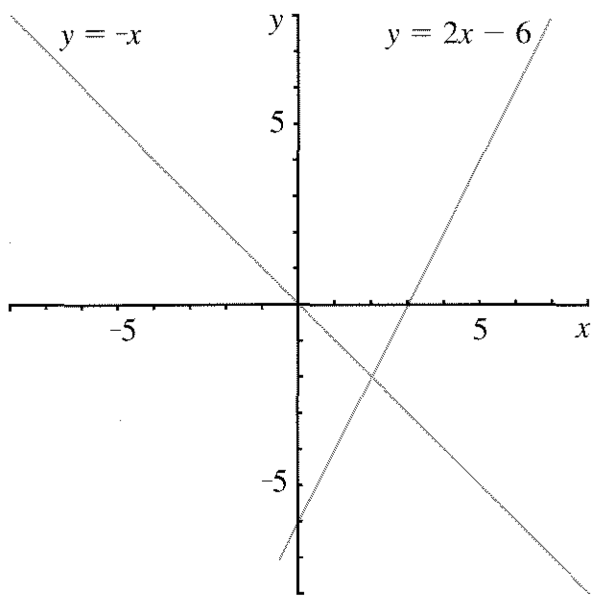
How Many Solutions?

You will need:

- graph paper
- graphing calculator (optional)

LINEAR EQUATIONS

As you learned in Chapter 6, graphing is one way to find solutions to equations. For example, consider the equation $2x - 6 = -x$. This equation can be solved by graphing the lines $y = 2x - 6$ and $y = -x$ on the same axes.



1. a. From the graph above, estimate the point of intersection of the lines $y = 2x - 6$ and $y = -x$.

b. Use algebra to solve the equation $2x - 6 = -x$.

2. The linear equation $2x - 6 = 2x$ has *no solution.* Show that this is true by graphing the lines $y = 2x - 6$ and $y = 2x$. Explain how your graph shows that the equation has no solution.

3. Tell how many solutions each equation has. Use graphs if necessary.

a. $5x - 6 = 5x - 7$

b. $5x - 6 = 0.5(10x - 12)$

c. $5x - 6 = x$

4. For all the equations in problem 3 that have one solution, find the solution.

5.

a. Write and solve a linear equation that has only one solution.

b. Write a linear equation that has an infinite number of solutions.

c. Write a linear equation that has no solution.

6. Is it possible for a linear equation to have two solutions? Three solutions? Explain your answers, using graphs if possible.

QUADRATIC EQUATIONS

Definition: Second-degree equations are called *quadratic equations.*

Example: These are all quadratic equations.

$x^2 = 45$

$3x^2 - 15 = 6x + 2$

$6x^2 + 5x + 8 = 0$

You will learn several methods for solving quadratic equations. In this lesson, we will use graphing. Use a whole piece of graph paper for problems 7-11.

7. Draw a pair of axes on a full page of graph paper. Show all four quadrants. Graph $y = x^2$ very carefully.

8. On the same pair of axes, graph these lines and label them with their equations.

a. $y = 6x - 12$

b. $y = 6x - 9$

c. $y = 6x - 5$

9. Label the point or points of intersection of each line with the graph of $y = x^2$.

One of the lines you drew touches the graph of $y = x^2$ at only one point.

Definition: A line that touches a graph at only one point is *tangent* to the graph.

10. Which of the lines you drew is tangent to the graph of $y = x^2$?

11. Use the graphs to solve these equations.
    a. $x^2 = 6x - 12$
    b. $x^2 = 6x - 9$
    c. $x^2 = 6x - 5$

HOW MANY INTERSECTIONS?

12. a. Draw a graph of $y = x^2$.
    b. On the same axes, draw a line that does not intersect $y = x^2$. Write the equation of the line.
    c. Repeat part (b) for another line that does not intersect $y = x^2$.

13. a. Draw a graph of $y = x^2$.
    b. On the same axes, draw a line that intersects $y = x^2$ at only one point. Write the equation of the line and label the point of intersection.
    c. Repeat part (b) for another line that intersects $y = x^2$ at only one point.

14. a. Draw a graph of $y = x^2$.
    b. On the same axes, draw a line that intersects $y = x^2$ at two points. Write the equation of the line and label the points of intersection.
    c. Repeat part (b) for another line that intersects $y = x^2$ at two points.

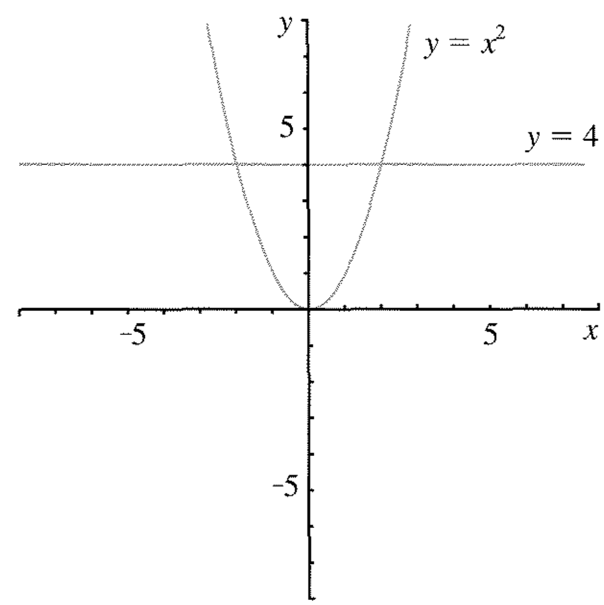
15. Refer to your answers to problems 12-14. Use them to write and solve a quadratic equation that has
    a. one solution;
    b. two solutions;
    c. no solutions.

16. Use graphs to estimate the solutions to these equations.
    a. $x^2 = -6x - 11$
    b. $x^2 = -6x + 11$
    c. $-x^2 = 6x + 11$

17. Write the equation of a line that is tangent to $y = x^2$ at the point (-4, 16).

WHICH GRAPH SHOULD YOU USE?

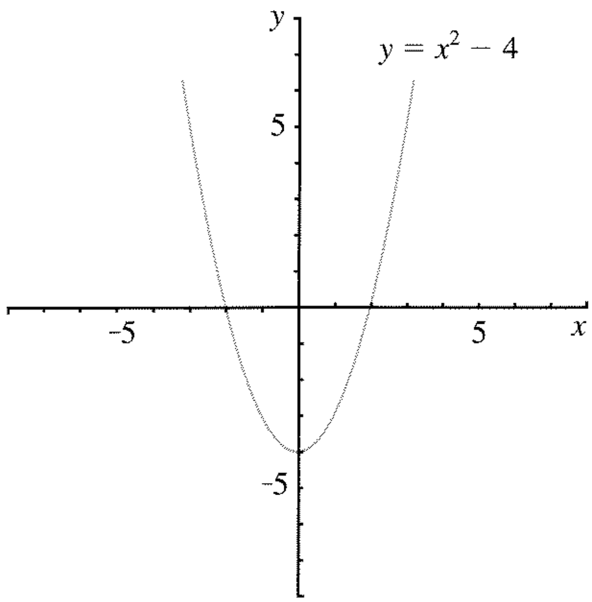
The solution of the equation $x^2 = 4$ can be found by graphing $y = x^2$ and $y = 4$ on the same pair of axes.



The equation $x^2 = 4$ can also be written as $x^2 - 4 = 0$. It can be solved by graphing $y = x^2 - 4$ and $y = 0$ on the same axes.

18. What is another name for the line $y = 0$?

As shown in the figure, the graphs intersect in two points. This means that the quadratic equation $x^2 = 4$ has two solutions.



19. What are the two values of x that satisfy the equation $x^2 = 4$? Where do they appear in each of the two graphs above?

20. Explain why all of these quadratic equations are equivalent.
$x^2 = x + 6$
$x^2 - x = 6$
$x^2 - x - 6 = 0$

21. Graph the parabola $y = x^2$ and the line $y = x + 6$ on the same pair of axes. Label the points of intersection.

22. Graph the parabola $y = x^2 - x$ and the line $y = 6$ on the same pair of axes. Label the points of intersection.

23. Graph the parabola $y = x^2 - x - 6$ and $y = 0$ on the same pair of axes. Label the points of intersection.

24. Compare your answers to problems 21-23.
   a. What is the solution to the quadratic equation $x^2 - x - 6 = 0$?
   b. Which of the three graphs do you think gave the easiest way to find the solution to this equation?

25. Find the solutions to these equations by graphing a parabola and a line on the same pair of axes. As you saw in problem 24, there may be more than one possible pair of graphs that can be used. You may use any pair that will work.
   a. $x^2 = 3x + 4$
   b. $x^2 - 5 = -4x$
   c. $2x^2 = 18$

DISCOVERY *LAST DIGITS*

26. What is the last digit for each of these numbers: $0^{100}, 1^{100}, 2^{100}, \ldots, 9^{100}$? Most of these numbers have too many digits for the last one to appear in your calculator, so you will have to figure out some other approach. (Hint: Try finding the last digits of smaller powers of these numbers.)

LESSON 7.7

Equations With Squares

You will need:

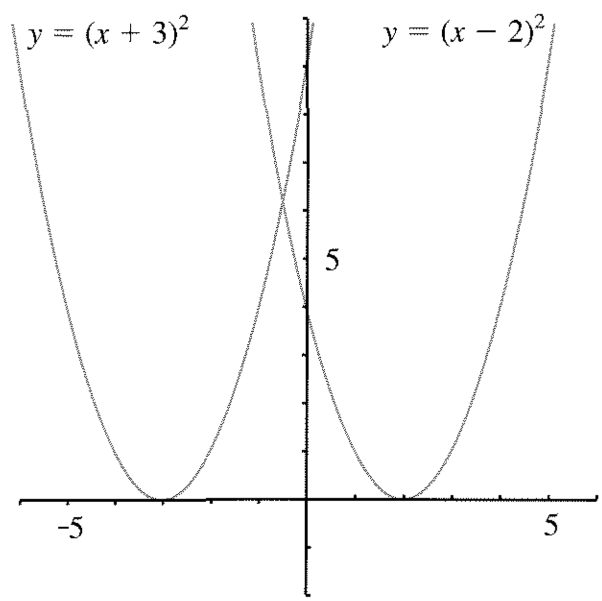
graph paper 

the Lab Gear 

This lesson is about solving equations. You will use two different methods to approach equations that involve the square of a binomial.

GRAPHICAL SOLUTIONS

The graphs of $y = (x + 3)^2$ and $y = (x - 2)^2$ are shown below.



1. Explain why these two graphs never go below the x-axis. (Why is the value of y never negative?)

2. On a piece of graph paper, copy the two graphs. For more accuracy, calculate the coordinates of several points on each curve. Use the graphs to solve these equations.
   a. $(x + 3)^2 = 4$
   b. $(x - 2)^2 = 9$
   c. $(x - 2)^2 = 1$
   d. $(x + 3)^2 = -1$
   e. $(x - 2)^2 = 0$

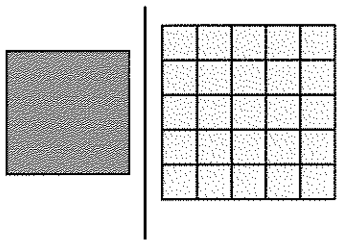
3. Use your graphs to estimate the solutions to these equations.
   a. $(x + 3)^2 = 12$
   b. $(x - 2)^2 = 6$
   c. $(x - 2)^2 = -2$
   d. $(x + 3)^2 = 5$
   e. $(x + 3)^2 = (x - 2)^2$

4.
   a. Describe what you think the graphs of the functions $y = (x + 2)^2$ and $y = (x - 1)^2$ would look like. (Where would each one intersect the x-axis?)
   b. Check your guess by making tables of values and graphing the functions.

5. Use your graphs to find or estimate the solutions to these equations.
   a. $(x + 2)^2 = 9$
   b. $(x + 2)^2 = 2x + 3$
   c. $(x - 1)^2 = 5$
   d. $(x - 1)^2 = -x$

EQUAL SQUARES

The equation $x^2 = 25$ can be illustrated using the Lab Gear. Put out your blocks like this.

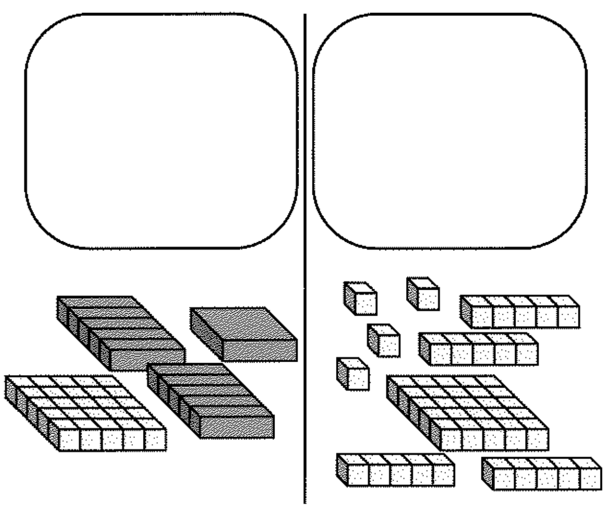


One way to get started with this equation is to remember that if the squares are equal, *their sides must be equal.* (This is true even though they don't look equal. Remember that x can have any value.)

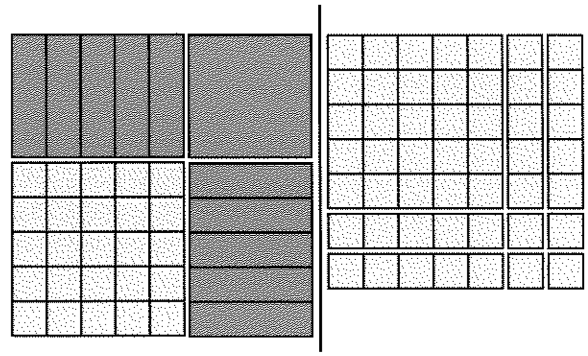
6. Solve the equation. If you found only one solution, think some more, because there are two.

7. Explain why there are two solutions.

8. Write the equation shown by this figure.



By rearranging the blocks, you can see that this is an *equal squares* problem, so it can be solved the same way. As you can see, $(x + 5)^2 = 7^2$. It follows that $x + 5 = 7$ or $x + 5 = -7$.



9. Solve the equation. There are two solutions. Check them both in the original equation.

Solve the equations 10-13 using the *equal squares* method. You do not have to use the actual blocks, but you can if you want to. Most equations, but not all, have two solutions.

10. $x^2 = 16$

11. $x^2 + 2x + 1 = 0$

12. $4x^2 = 36$

13. $4x^2 + 4x + 1 = 9$

Solve these equations without the blocks.

14. $4x^2 - 4x + 1 = -9$

15. $x^2 - 10x + 25 = 16$

16. $x^2 + 6x + 9 = 4x^2 - 4x + 1$

17. $-x^2 - 6x - 9 = -25$ (Hint: If quantities are equal, their opposites must be equal.)

18. Explain why some problems had one, or no solution.

COMPARING METHODS

19. Summary

a. Compare the graphical method and the Lab Gear method for the solution of an equal-squares equation. Use examples that can be solved by both methods, and have 0, 1, and 2 solutions.

b. What is the meaning of the x-intercept in the graphical method? Where does that number appear in the Lab Gear method?

c. What is the meaning of the x- and y-coordinates of the intersections of the line and parabola in the graphical method? Where do these numbers appear in the Lab Gear method?

20. Create an equal-squares equation that has two solutions that are not whole numbers. Solve it.

REVIEW *FACTORING PRACTICE*

Factor these polynomials. One is difficult, one is impossible. The Lab Gear may help for some of the problems.

21. $xy + 6y + y^2$ **22.** $y^2 - 16$

23. $3x^2 + 13x - 10$ **24.** $4x^2 + 8x + 4$

25. $2x^2 + 2x + 1$ **26.** $y^2 - 5y + 6$

27. $y^2 - 4y + 4$ **28.** $x^2 + 8x + 12$

REVIEW *MULTIPLICATION PRACTICE*

You can multiply polynomials without the Lab Gear and without a table. Picture the table in your mind, and make sure you fill all its spaces. For example, to multiply

$$(2 - x)(7 - 3x + 5y)$$

you would need a 2-by-3 table. To fill the six cells of the table, you would multiply the 2 by 7, by $-3x$, and by $5y$. Then you would multiply the $-x$ by 7, by $-3x$, and by $5y$. Finally, you would combine like terms. While you think of the six cells of the table, what you actually write on paper looks like this.

$$(2 - x)(7 - 3x + 5y)$$
$$= 14 - 6x + 10y - 7x + 3x^2 - 5xy$$
$$= 14 - 13x + 10y + 3x^2 - 5xy$$

29. Look at the example above, and make sure you understand where each term came from.

Multiply these polynomials without using a table. Combine like terms.

30. a. $(2x - y)(y + 3x)$
b. $(x - 5y)(3x + 2y)$
c. $(ac - b)(2b + 2ac)$
d. $(ab - c)(b - c^2)$

31. a. $(2x - y + 4)(5 - y)$
b. $(2x^2 - y + 4)(5y - x^2)$
c. $(2x - y^2 + 4)(5y - x^2)$
d. $(a + b + c)(2a + 3b + 4c)$

LESSON

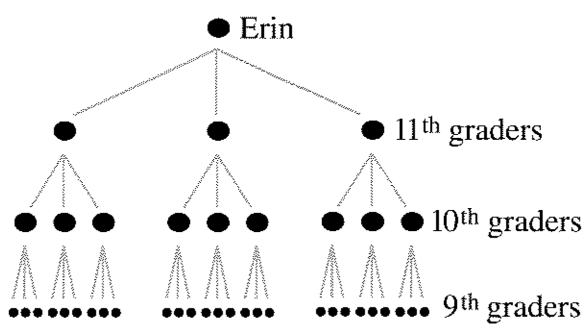
7.8

Power Play

RAFFLE TICKETS

Erin is a senior at Alaberg High School and the director of the senior class play. To help pay for sets and costumes, she plans to raise money through a raffle. She is considering several plans for selling raffle tickets.

Erin's first idea was to have members of each class sell raffle tickets to the class below them. Erin would sell tickets to three 11th graders. Each of them would sell tickets to three 10th graders, who in turn would each sell tickets to three 9th graders, and so on. Erin started to draw a tree-diagram of her plan.



1. If Erin extended her plan all the way down to first grade, how many first graders would be buying tickets? Explain.

2. Make a table like the one following showing how many tickets would be bought by students in each grade. (The first entry in the table is based on the assumption that Erin bought one ticket for herself.) In the last column, express the number of tickets as a power of 3.

Grade	Tickets (number)	Tickets (as a power of 3)
12th	1	
11th	3	3^1
...		
1st		

3. Give several reasons why Erin's plan is not practical.

THE EXPONENT ZERO

The last column in your table above contained increasing powers of 3.

4. a. To follow the pattern, what should the exponent on the first power in the table be?

b. Based on that pattern, what should 3^0 be equal to?

5. a. Copy and complete this table.

5^5	3125
5^4	
5^3	
5^2	
5^1	

b. As you move down the columns, how can you get the next row from the previous one?

c. Add another row to the bottom of the table. Explain how it fits the pattern.

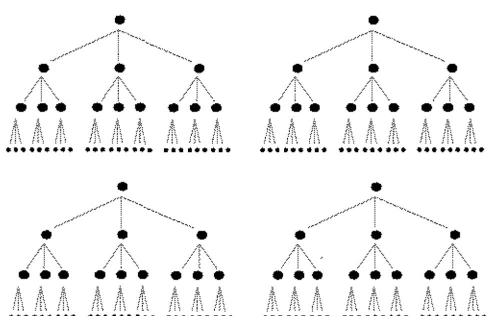
6. **Generalization** You have found the values of 3^0 and 5^0. Using patterns in the same way, find the values of 2^0 and 4^0. What generalization can you make?

7. **Summary** Many people think that a number raised to the zero power should be zero. Write a few sentences explaining why this is not true.

A BETTER PLAN

Erin needs a better scheme for selling raffle tickets. She decides to enlist the help of other seniors in the play. Each senior (including Erin) will buy a ticket for himself or herself, and sell a ticket to three juniors; each of the juniors will sell a ticket to three sophomores; and so on, down to the 8th grade.

Four more seniors help out:



8. Assume Erin gets five seniors to help (including herself).
   a. How many 8th graders would buy tickets?
   b. How is this number related to the number of 8th graders who would buy tickets if Erin does not get any other seniors to help?
   c. Express the answer to (a) as a number times a power of 3. Explain.

9. If Erin gets K seniors to help (including herself), how many 8th graders would buy tickets? Express the answer in terms of K.

10. Assume five seniors are involved, including Erin. As before, each student at every step buys one ticket, but now each student sells two tickets instead of three.
    a. How many 8th graders would buy tickets?
    b. Express the answer to (a) as a number times a power. Should you use a power of 2, a power of 3, or a power of 5? Explain your answer.

11. Assume K seniors are involved and each student sells M tickets.
    a. How many 8th graders would buy tickets? Express your answer in terms of K and M.
    b. How many Nth graders would buy tickets? Express your answer in terms of K, M, and N.

12. **Exploration** Erin hopes to sell 1500 tickets altogether. Find several values for K (the number of seniors) and M (the number of tickets sold per person) that make it possible to sell at least 1500 tickets, without going below 7th grade. For each plan, indicate the number of students who would be involved at each grade level. Which of those plans do you think is the most realistic?

REVIEW *WHICH IS GREATER?*

13. Which is greater?
a. $5 \cdot 3^{35}$ or $3 \cdot 5^{35}$
b. $5 \cdot 30^{35}$ or $30 \cdot 5^{35}$
c. $5 \cdot 300^{35}$ or $300 \cdot 5^{35}$

14. Which is greater?
a. $5^{35} \cdot 3^{35}$ or 15^{35}
b. 35^0 or 0^{35}

15. If a and b are each greater than 1, which is greater, $(ab)^{10}$ or ab^{10}? Explain.

REVIEW *A COMMUTATIVE LAW?*

Al announced, "I noticed that $4^2 = 2^4$ and $3^2 = 2^3$, so I generalized this using algebra to say $a^b = b^a$, always."

"That's a great discovery," said Beau. "This means that exponentiation is commutative!"

"Nice try, Al," said Cal. "It's true that 4^2 and 2^4 are both 16, but 3^2 is 9 and 2^3 is 8. They aren't equal."

Al was disappointed. "Round-off error," he muttered. "Close enough."

16. What did Beau mean when she said that exponentiation is commutative? Is she right or wrong? Explain, using examples to support your answer.

17. Is $4^2 = 2^4$ the only case where $a^b = b^a$? If it is, how can you be sure? If it isn't, how can you find others?

18. **Exploration** Which is greater, a^b or b^a? Of course, the answer to this question depends on the values of a and b. Experiment, and try to make some generalizations.

REVIEW/PREVIEW *CHUNKING*

19. Solve for y: $y^2 = 49$. (Remember there are two solutions.)

You can use the strategy of chunking to solve equations involving squares. For example, in problem 20, think of $(x + 3)$ as a chunk, and write two linear equations.

Solve.

20. $(x + 3)^2 = 49$

21. $(2p - 5)^2 = 49$

22. $(5 - 2p)^2 = 49$

23. $(6 + 2r)^2 = 49$

THINKING WRITING

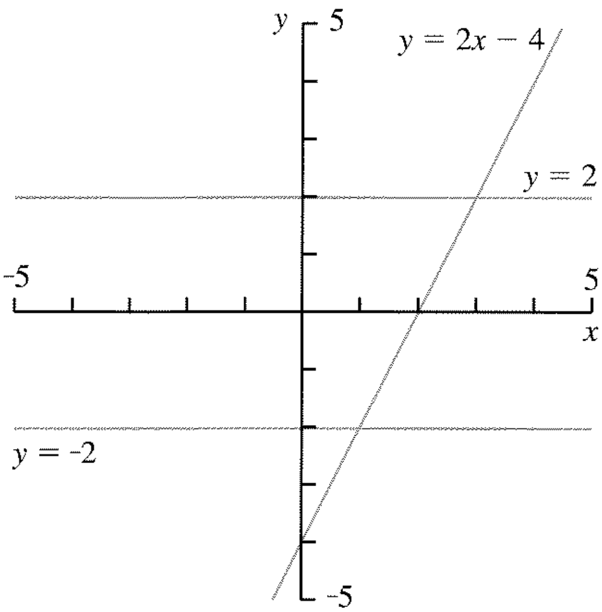
7.B Graphing Inequalities

COMPOUND INEQUALITIES

Definition: An inequality that contains more than one inequality symbol is called a *compound inequality.*

Example: $3 < 2x < 8$ is read *2x is between 3 and 8.*

The figure shows the graphs of the line $y = 2x - 4$ and the horizontal lines $y = 2$ and $y = -2$.



1. What are the coordinates of the points of intersection of $y = 2x - 4$ with each of the horizontal lines?

2. Look only at the part of the line $y = 2x - 4$ that is between the lines $y = 2$ and $y = -2$.
   a. Give the coordinates of some of the points on this part of the line.
   b. On this part of the line, how large can the y-coordinate get? How small?
   c. On this part of the line, how large can the x-coordinate get? How small?

We say that the *solution* of the compound inequality $-2 < 2x - 4 < 2$ is

$$1 < x < 3.$$

Notice that the solution is also a compound inequality, but it is simpler than the original one. It tells us what values of x make the first inequality true.

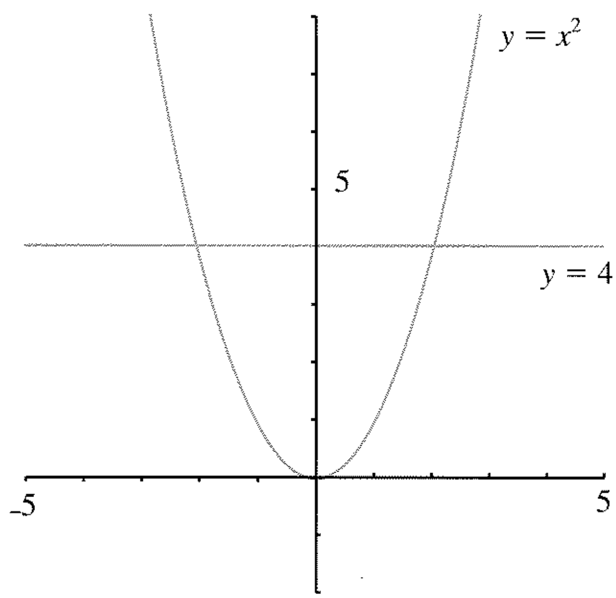
3. Explain how the graph above can be used to show that the solution to the inequality is $1 < x < 3$.

4. a. Graph the horizontal lines $y = 3$, $y = 8$, and $y = 3x + 5$.
   b. Use your graph to find the solution of the compound inequality

$$3 < 3x + 5 < 8.$$

QUADRATIC INEQUALITIES

Sometimes an inequality is not compound, but it has a compound solution. An example is the inequality $x^2 < 4$. The two graphs shown can be used to solve this inequality.



5. Look at the part of the graph of $y = x^2$ that is below the graph of $y = 4$.

a. Give the coordinates of four points that lie on this part of the graph.

b. On this part of the curve, how large can the x-coordinate get? How small?

c. Write the solution to this inequality.

6. The same graph can also be used to solve the inequality $x^2 > 4$. In this case, the solution cannot be written as a compound inequality. Instead it is written in two parts,

$$x < -2 \text{ or } x > 2.$$

Explain why the solution has two parts.

7. On the same pair of axes, make an accurate graph of $y = x^2$, $y = 1$, and $y = 9$. Use your graphs to solve these inequalities.

a. $x^2 < 9$ b. $x^2 > 9$

c. $x^2 < 1$ d. $x^2 > 1$

e. $1 < x^2 < 9$

8. Use the graph to estimate the solution to

$$x^2 > 5.$$

9. Solve these without a graph.

a. $x^2 < 16$ b. $x^2 > 16$

c. $x^2 > 0$ d. $x^2 < 0$

10. Report Write an illustrated report summarizing what you have learned in this assignment. Use examples, including at least one quadratic, and at least one compound, inequality.

LESSON 7.9

Powers and Large Numbers

Powers provide a shorthand for writing large numbers. Just as multiplication is repeated addition, raising to a power is repeated multiplication. For example,

$$12 \cdot 12 \cdot 12 \cdot 12 \cdot 12 \cdot 12,$$

which equals 2,985,984, can be written 12^6. Not only is this shorter to write than either the repeated multiplication or the decimal number, it is shorter to key into a scientific calculator.

Notation: Calculators use [^], [x^y], or [y^x] for *exponentiation* (raising to a power). We will use [^] to refer to that key.

Calculators can calculate with exponents that are not positive whole numbers. For example, it is possible to get a value for a number like $3^{-2.4}$ using the key for powers on your calculator. (Try it.) In this lesson, you will consider only *positive whole numbers* for exponents. In later chapters, you will use other exponents.

APPROXIMATING LARGE NUMBERS

1. **Exploration** Consider the number 123,456. Use your calculator to approximate the number as closely as you can with a power of 2, a power of 3, a power of 9, and a power of 10. How close can you get with each power? Repeat this experiment with four other numbers. (Use the same numbers as other students, so as to be able to compare your answers. Use only positive whole numbers having six or more digits.) Is it possible to get close to most large numbers by raising small numbers to a power?

For problems 2-4 find the powers of the following numbers (a-d), that are immediately below and above the given numbers.

a. 2 b. 3
c. 9 d. 10

Example: 691,737 (the population of Virginia, the most heavily populated state in 1790) is:

a. between 2^{19} and 2^{20}
b. between 3^{12} and 3^{13}
c. between 9^6 and 9^7
d. between 10^5 and 10^6

2. 3,929,214 (the population of the United States in 1790)

3. 48,881,221 (the number of people who voted for President Bush in 1988)

4. 178,098,000 (approximate number of Americans aged 18 or older in 1988)

CLOSER APPROXIMATIONS

It is possible to combine powers with multiplication to get approximations that are closer than those you were able to get in the previous sections by using only powers. For example, the speed of light is approximately 186,282 miles per second. This number is more than 2^{17} and less than 2^{18}, since

$$2^{17} = 131{,}072 \text{ and } 2^{18} = 262{,}144.$$

By multiplying 131,072 by a number less than 2, it is possible to get quite close to 186, 282.

$1.2 \cdot 131{,}072 = 157{,}286.4$ (too small)
$1.5 \cdot 131{,}072 = 196{,}608$ (too large)
$1.4 \cdot 131{,}072 = 183{,}500.8$ (too small, but pretty close)

5. We showed that the speed of light can be roughly approximated by multiplying 2^{17} by 1.4. Find an even better approximation by changing the number by which you multiply, using more places after the decimal point.

You can approximate the speed of light in many different ways using powers of 2. For example:

$93141 \cdot 2^1 = 186{,}282$
$46570 \cdot 2^2 = 186{,}280$
$23280 \cdot 2^3 = 186{,}240$
$45.5 \cdot 2^{12} = 186{,}368$

We used 2^{17} in the example instead of some other power of 2 because it is the *largest* power of 2 that is less than 186,282. We approximated 186,282 by *multiplying that power of 2, by a number between 1 and 2.*

6. Write an approximation to the speed of light using a power of 3 multiplied by a number between 1 and 3. (Hint: Begin by finding the largest power of 3 that is less than 186,282.)

7. Write an approximation to the speed of light using a power of 9 multiplied by a number between 1 and 9.

8. Write an approximation to the speed of light using a power of 10 multiplied by a number between 1 and 10.

9. Combine a power and multiplication to get a close approximation to the length of the Earth's equator, which is 24,902 miles, to the nearest mile. You can use any base for the power, but multiply the power by a number between 1 and the base.

NAMES FOR LARGE NUMBERS

10. Write 100 and 1000 as powers of 10.

There are common names for some of the powers of ten. *Billion* in the U.S. means 10^9, but in Britain it means 10^{12}. The table gives the common names used in the U.S. for some powers of ten.

Power	Name
10^6	million
10^9	billion
10^{12}	trillion
10^{15}	quadrillion
10^{18}	quintillion
10^{21}	sextillion
10^{100}	googol

11. Someone might think a billion is two millions, and a trillion is three millions. In fact, a billion is how many millions? A trillion is how many millions? Explain.

SCIENTIFIC NOTATION

Definition: To write a number in *scientific notation* means to write it as a power of 10 multiplied by a number between 1 and 10. This is the most common way of writing large numbers in science and engineering.

12. Explain why 10 is used for the base in scientific notation rather than some other number. Use examples.

13. Write in scientific notation.

a. one million
b. 67 million (the average distance from the sun to Venus in miles)
c. 5.3 billion (an estimate of the world's population in 1990)
d. twenty billion
e. 3.1 trillion (the U.S. national debt in dollars as of June 1990)
f. three hundred trillion

14. **Project** Find four large numbers that measure some real quantity. They should all be larger than 100,000,000. Encyclopedias, almanacs, and science books are good sources of such numbers.

a. Tell what each number measures.
b. Write the number in scientific notation.

REVIEW *PRIME NUMBERS*

There is only one polyomino rectangle of area 2, and only one of area 3. But there are two polyomino rectangles of area 4, corresponding to the products $2 \cdot 2 = 4$ and $1 \cdot 4 = 4$.

15. What is the smallest number that is the area of a polyomino rectangle in 3 different ways? Sketch the three rectangles and show the products.

16. Repeat the problem for 4 different ways.

17. Can you predict the smallest number that is the area of a polyomino rectangle in 5 different ways? Check your prediction.

Definition: Numbers greater than 1 that can only make a rectangle with whole number dimensions in one way are called *prime numbers*.

18. Here is an ancient method, (invented by the Greek mathematician Eratosthenes,) of finding the prime numbers.

a. On a list of numbers from 1 to 100, cross out the 1.
   Circle 2, cross out its multiples.
   Circle the first number that is not crossed out, cross out its multiples.
   Repeat, until all the numbers are either crossed out or circled.
b. Explain how and why this method works to find the prime numbers.

19. A mathematician once suggested that *every even number greater than 2 may be the sum of two prime numbers.* No one knows why this should be true, but it has worked for every number that's ever been tried. Test this for yourself with at least ten even numbers. (This is known as *Goldbach's conjecture.* A conjecture is a guess that has not yet been proved true or false.)

LESSON 7.10

Using Scientific Notation

WITH A CALCULATOR

Calculators can display numbers only up to a certain number of digits. For many calculators, ten digits is the limit.

1. What is the limit for your calculator?
2. What is the smallest power of 2 that forces your calculator into scientific notation?

On many calculators, the answer to problem 2 is 2^{34} which, according to the calculator, is equal to

[1.717986918^{10}] or [1.717986918E10].

The expression on the left does *not* mean 1.717986918^{10}, even though that's what it looks like. It is just calculator shorthand for $1.717986918 \cdot 10^{10}$. The actual value is 17179869184, which is too long to fit, so the calculator gives the approximate value of 17179869180, expressed in scientific notation. (For a number this large, this represents a very small error.)

3. Which power of 2 is displayed as [2.814749767E14] ?
4. Find a power of 4 and a power of 8 that are also displayed as [2.814749767E14].
5. Find powers of 3, 9, 27, and 81 that are displayed in scientific notation, in the form ____ $\cdot 10^{17}$. If possible, find more than one solution for each number.

There are three ways to enter numbers in scientific notation into your calculator. For example, to enter $2 \cdot 10^3$, you can key in 2 [*] 10 [^] 3, or 2 [*] [10^x] 3, or (depending on the calculator) 2 [EE] 3, or 2 [EXP] 3. We will refer to this last key as [EE].

6. Try all the methods listed that are available on your calculator. In each case, the calculator should respond with [2000] after you press [=] or [ENTER].
7. Explain the purpose of the [^] and [EE] keys. How are they different?

HOW MUCH FARTHER, HOW MANY TIMES AS FAR?

The table shows the ten brightest objects in the sky, and their *average* distances from Earth, in miles. (The objects are listed in order of average brightness as seen from Earth.)

	Distance
Sun	$9.29(10^7)$
Moon	$2.39(10^5)$
Venus	$9.30(10^7)$
Jupiter	$4.84(10^8)$
Sirius	$5.11(10^{13})$
Canopus	$5.76(10^{14})$
Arcturus	$2.12(10^{14})$
Mars	$1.42(10^8)$
Vega	$1.59(10^{14})$
Saturn	$8.88(10^8)$

8. If you were to divide the objects into two groups, based only on the value of the exponents of 10, what would be in each group? What is the actual significance of the two groups?

For each pair of objects given in problems 9-13, answer questions (a) and (b). If an answer is greater than 10,000, give it in scientific notation.

a. The second object is *how many miles* farther from Earth than the first?

b. The second object is *how many times* as far from Earth as the first?

9. The Moon, Venus

10. The Moon, Saturn

11. The Sun, Sirius

12. The Sun, Canopus

13. Sirius, Canopus

WITHOUT A CALCULATOR

14. Convert these numbers to ordinary decimal notation and add them without a calculator.

a. $(4 \cdot 10^7) + (5 \cdot 10^6)$

b. $(40 \cdot 10^6) + (5 \cdot 10^6)$

15. Compare the two computations in problem 14. Which would have been easy to do without converting to ordinary decimal notation? Explain.

Without a calculator it is not easy to add and subtract in scientific notation. One way is to revert to ordinary decimal notation. Another is to write the two quantities with a common exponent for 10, as was done in problem 14b.

16. Add or subtract.

a. $6.2 \cdot 10^3 + 5 \cdot 10^6$

b. $6.2 \cdot 10^6 - 5 \cdot 10^3$

c. $6.2 \cdot 10^5 + 5 \cdot 10^3$

d. $6.2 \cdot 10^3 - 5 \cdot 10^6$

Without a calculator it can be tedious to multiply and divide large numbers. However, if the numbers are written in scientific notation it is easy to estimate the size of the answer.

For the following problems, 17-20:

a. Convert the numbers to ordinary decimal notation.

b. Multiply or divide.

c. Write your answers in scientific notation.

17. $(3 \cdot 10^5) \cdot (6 \cdot 10^3)$

18. $(3 \cdot 10^3) \cdot (6 \cdot 10^5)$

19. $(6 \cdot 10^6) \div (3 \cdot 10^3)$

20. $(3 \cdot 10^6) \div (6 \cdot 10^3)$

PREVIEW *MULTIPLICATION AND EXPONENTS*

21.

a. In each of problems 17-20, look for a relationship between your answer and the original numbers. How could you have obtained your answer without converting from scientific notation?

b. Explain a shortcut for multiplying and dividing numbers in scientific notation. Include an explanation of what happens to the exponent of 10.

22. Does the shortcut, described in problem 21b, work for multiplying $3(2^4)$ by $5(2^6)$? Explain, giving several examples of this type.

REVIEW *PERFECT SQUARE TRINOMIALS*

23. All of these are perfect square trinomials. Write each one as the square of a binomial.

a. $c^2x^2 + 2bcxy + b^2y^2$

b. $y^2 + 2xy + x^2$

c. $y^2 + 2by + b^2$

d. $0.25x^2 + 0.2x + 0.04$

LESSON 7.11

Using Large Numbers

TRAVELING IN THE SOLAR SYSTEM

The table below gives the diameter and average distance from the Sun in kilometers (km) of each of the planets in the solar system. The Sun's diameter is also shown.

	Diameter	Distance from Sun	Moons
Sun	$1.39(10^6)$		
Mercury	$4.88(10^3)$	57,700,000	0
Venus	$1.21(10^4)$	108,150,000	0
Earth	$1.23(10^4)$	150,000,000	1
Mars	$6.79(10^3)$	227,700,000	2
Jupiter	$1.43(10^5)$	778,300,000	17
Saturn	$1.20(10^5)$	1,427,000,000	22
Uranus	$5.18(10^4)$	2,870,000,000	15
Neptune	$4.95(10^4)$	4,497,000,000	3
Pluto	$6.00(10^3)$	5,900,000,000	1

1. Convert the diameters to normal decimal notation.
2. Convert the distances to scientific notation.
3. Divide the planets into groups according to:
   a. their diameters. How many groups are there? Explain.
   b. their distance from the Sun. How many groups are there? Explain.
   c. their number of moons. How many groups are there? Explain.
4. Compare the groups you created in problem 3. Find a way to combine your decisions into an overall division of the planets into two or three groups, by *type of planet*. Name each group, and list its characteristics in terms of the data in the table.
5. Light travels approximately 299,793 kilometers per second. Show your calculations, and give your answers in scientific notation. How far does light travel in
   a. one minute? b. one hour?
   c. one day? d. one year?
6. Abe remembers learning in elementary school that it takes about eight minutes for light to travel from the Sun to the Earth. Figure out whether he remembers correctly. Show your calculations.
7. Light from the Sun takes more than one day to reach which planets, if any?
8. When Pluto is at its mean distance from the Sun, how long does it take light from the Sun to reach it?
9. An *Astronomical Unit* is the distance from the Earth to the Sun. What is Pluto's distance from the Sun in Astronomical Units?

SCALE MODELS

10. Make a scale drawing showing the distances of the planets from the Sun. Tell what your scale is, and explain why you chose it.

Project

11. Decide what would be a good scale for a scale model of the solar system, so you could fit the model in your classroom. How large would each planet be? How far would each planet be from the Sun?

12. Decide what would be a good scale for a scale model of the solar system, so you could clearly see even the smallest planet. How far would each planet be from the Sun? How large would each planet be? What objects could you use to represent the planets?

13. Using a map of your town, figure out where you might place the planets and the Sun. Use the scale you calculated in problem 12.

14. The nearest star, Alpha Centauri, is 40 trillion kilometers away from the Sun. Where would it be in your model?

DOWN TO EARTH

15. In 1986 people in the U.S. threw away about 64.7 million tons of paper and cardboard. Write this number in scientific notation.

The number 64.7 million is too large to mean anything to most people. The following problems illustrate some ways of bringing large numbers "down to earth."

For example, to understand how much paper and cardboard was thrown away in the U.S. in 1986, it helps to figure out how much was thrown away *per person.*

Since there are 2000 pounds in a ton, 64.7 million tons is

$(6.47 \cdot 10^7 \text{ tons}) \cdot (2 \cdot 10^3 \text{ lbs/ton})$
$= 12.94 \cdot 10^{10} \text{ lbs.}$

16. The U.S. population in 1986 was about 240 million people. Write this number in scientific notation. Then calculate how many pounds of paper and cardboard were thrown away *per person.*

17. The distance around the equator of the Earth is about 24,900 miles. Al bikes to and from school every day, about five miles each way. Biking back and forth to school, about how many school years would it take Al to cover the distance around the equator? (A school year has about 180 days.)

18. Biking back and forth to school, about how many school years would it take Al to cover
    a. the distance from the Earth to the moon?
    b. the distance from the Earth to the Sun?

19. The population of the U.S. was about 250 million in 1990. Approximately $5 \cdot 10^{11}$ cigarettes were smoked in the U.S.
    a. About how many cigarettes were smoked *per person*?
    b. About how many were smoked *per person, per day*?
    c. If 186 million U.S. residents did not smoke any cigarettes, how many cigarettes were smoked *per smoker, per day*?

LESSON 7.12

As the Crow Flies

You will need:

- geoboards
- dot paper

SQUARE ROOTS

As you know, the square of a number is the area of a square that has that number for a side. For example, the square of 4 is 16, because a square having side 4 has area 16.

1. a. What is the area of a square having side 9?

b. What is the side of a square having area 9?

2. a. What is the area of a square having side 10?

b. What is the side of a square having area 10?

You can answer question 2b with the help of a calculator, by using trial and error. Or, you may answer it by using the $\boxed{\sqrt{}}$ key.

Definition: The *square root* of a number is the side of a square that has that number for area.

For example, the square root of 4 is 2, because a square having area 4 has side 2.

3. a. What is the square of 11?

b. What is the square root of 11?

The square root of 11 is written $\sqrt{11}$. The number given by a calculator is an approximation of the exact value. Many calculators have an $\boxed{x^2}$ key.

4. Use the $\boxed{x^2}$ key to calculate the square of 8.76. Write it down. Clear your calculator. Now use the $\boxed{\sqrt{}}$ key to find the square root of the number. What answer did you get? Explain why this is so.

5. Find a number for $\sqrt{5}$. Write it down. Now clear your calculator, enter the number, and use the $\boxed{x^2}$ key. What answer did you get? Compare your answer with other students' answers. Explain.

6. Which number has more digits, $\sqrt{10.3041}$ or $\sqrt{2}$? Make a prediction and check it with your calculator. Explain your answer.

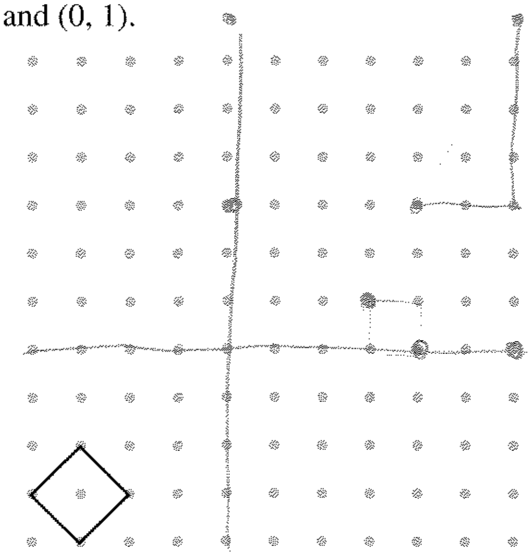
DISTANCE ON THE GEOBOARD

To find the distance between two points on the geoboard, *as the crow flies,* you can use the following strategy.

- Make a square that has the two points as consecutive vertices.
- Find the area of the square.
- Find the side of the square.

In problems 7-9, express your answers two ways: as a square root, and as a decimal approximation (unless the answer is a whole number).

Example: Find the distance between (1, 0) and (0, 1).



The area of the square is 2, so the distance between the two points is $\sqrt{2}$, or 1.41...

7. Find the distance between:

a. (4, 3) and (6, 7);

b. (4, 6) and (6, 4);

c. (4, 5) and (4, 8).

8. Find the distance between the origin and (3, 1).

9. Find the distance between (5, 5) and (8, 9).

10. a. Find 12 geoboard pegs that are at a distance 5 from (5, 5). Connect them with a rubber band. Sketch the figure.

b. Explain why someone might call that figure a *geoboard circle.*

11. How many geoboard pegs are there whose distance from (5, 5) is

a. greater than 5?

b. less than 5?

12. Choose a peg outside the *circle* and find its distance from (5, 5).

13. Find all the geoboard pegs whose distances from (4, 3) and (6, 7) are equal. Connect them with a rubber band. Sketch.

14. What are the distances between the pegs you found in problem 13 and (4, 3) or (6, 7)?

15. **Generalization** Describe a method for finding the distance between the origin and a point with coordinates (x, y). Use a sketch and algebraic notation.

DISCOVERY SUMS OF PERFECT SQUARES

16. Any whole number can be written as a sum of perfect squares. Write each whole number from 1 to 25 as a sum of squares, using *as few squares as possible* for each one. (For example, $3^2 + 1^2$ is a better answer for 10 than $2^2 + 2^2 + 1^2 + 1^2$.)

17. You should have been able to write every number in problem 16 as a sum of *four or fewer* perfect squares. Do you think this would remain possible for large numbers? For very large numbers? Experiment with a few large numbers, such as 123, or 4321.

DISCOVERY SUMS OF POWERS

18. Write every whole number from 1 to 30 as a sum of powers of 2. Each power of 2 cannot be used more than once for each number. Do you think this could be done with very large numbers? Try it for 100.

19. Write every whole number from 1 to 30 as a sum of powers of 3 and their opposites. Each power can appear only once for each number. Do you think this could be done with very large numbers? Try it for 100.

THINKING WRITING

7.C One Googol Zeroes

Scientific notation will help you think about these two very large numbers.

one googol = 10,000,000,000,000,000,000,
000,000,000,000,000,000,000,000,000,000,
000,000,000,000,000,000,000,000,000,000,
000,000,000,000,000,000,000

1. How many zeroes does it take to write one googol? (Count them!)

one googolplex = 1 followed by one googol zeroes

2. Guess how large a sheet of paper one would need to write one googol zeroes.

a. a sheet the size of a table?

b. a sheet the size of a room?

c. a sheet the size of a school?

d. a sheet the size of a city?

3. Let's assume a zero takes up one square centimeter. How many zeroes could you fit on a piece of paper having area

a. one square meter? (There are 100 centimeters in a meter. Use a sketch to figure out how many square centimeters in a square meter. Hint: There are more than 100 square centimeters in a square meter.)

b. one square kilometer? (There are 1000 meters in a kilometer.)

Notation: cm^2 stands for square centimeter; km^2 for square kilometer.

4. a. The area of California is $4(10^5)$ km^2. How many zeroes could fit on a sheet of paper this size?

b. The area of the United States is nearly 10^7 km^2. How many zeroes could fit on a sheet this size?

5. 30,000 sheets of thin paper make a pile one meter high. How many zeroes could be in such a pile, if each sheet is the size of the United States?

6. a. The moon is less than $4(10^5)$ km away. How many zeroes, if our pile of paper extended that far?

b. The sun is $1.5(10^8)$ km away. How many zeroes, if our pile extended that far?

c. The nearest star is $4(10^{13})$ km away. How many zeroes, if our pile extended that far?

7. What fraction of the total number of zeroes does our pile include?

8. Report Write a report summarizing your answers to problems 3-6 above. Show your calculations and include any sketches that were useful in figuring out answers. Explain your reasoning. Then show how to figure out the correct answer to problem 2.

9. Project Where in the universe would our pile of papers end if it did include one googol zeroes?

The word *googol* was created in 1938 by the eleven-year-old nephew of the American mathematician Edward Kasner. In one sense, a googolplex is the largest number that has a name. But in fact, even without creating any new names, you can name larger numbers. For example, the words *two googolplex* name a larger number.

10. What is the largest nameable number? Explain your reasoning.

Essential Ideas

RECTANGULAR WINDOWS

The window panes referred to below are those pictured in Lesson 2 of this chapter.

1. Sketch a window having length equal to twice its width that is made up of panes from the A.B. Glare Co. How many panes of each type (corner, edge, and inside) are there?

Use sketches or tables of values to help solve the following problem.

2. How many of each type of pane would you need for windows that are twice as long as they are wide? Your answer will depend on the width of the window. Let the width be W, and find expressions in terms of W for:

a. the length of the window;
b. the number of inside panes;
c. the number of edge panes;
d. the number of corner panes.

3. Draw a pair of axes and label the x-axis *Width* and the y-axis *Number of Panes.* Then make a graph showing each of these as a function of W, the width of the window.

a. the number of inside panes
b. the number of edge panes
c. the number of corner panes

4. As you increase the width of the window, which grows fastest, the number of inside panes, edge panes, or corner panes? Explain, referring to graphs or sketches.

5. The panes described in Lesson 2 cannot be used for windows of width 1.

a. Explain why.
b. Sketch the two types of panes that are needed in this case.
c. Find the number of each type of pane for a window having width 1 and length L.

MULTIPLY

6. Multiply these polynomials.

a. $(3x - 5)(4x - 6)$
b. $(5 - 3x)(6 - 4x)$
c. $(3y + 3x - 1)(-2x + 2y)$
d. $(x + y + z)(-x + y)$

7. Multiply and compare the results. What do you notice? Explain.

a. $(ax - by)(ax - by)$
b. $(by - ax)(by - ax)$
c. $(ax - by)(by - ax)$

REMARKABLE IDENTITIES

8. Find the missing terms.

a. $(ax - \_\_\_\_)^2 = \_\_\_\_ - 2ax + 1$
b. $b^2 - x^2 = (b - x)(\_\_\_\_)$
c. $y^2 - 10y + \_\_\_\_ = (\_\_\_\_)^2$
d. $(ax + \_\_\_\_)^2 = a^2x^2 + \_\_\_\_ + b^2y^2$

FACTOR

Factor. Look for a common factor and use an identity.

9. a. $5x^2 + 20x + 20$
   b. $6y^2 + 12xy + 6x^2$
   c. $2x^2 + 60x + 450$

10. Find the middle term that will make each of these a perfect square trinomial. Then write it as the square of a binomial.
    a. $100a^2 +$ ____ $+ 49b^2$
    b. $(1/9)x^2 +$ ____ $+ (1/4)y^2$

11. Factor these polynomials.
    a. $4x^2 - 20x + 25$
    b. $4x^2 - 25$
    c. $25 - 4x^2$

12. Factor these polynomials. (Hint: First look for common factors.)
    a. $5y^2 + 90xy + 45x^2$
    b. $48x - 27xy^2$
    c. $xy^2 - 6x^2y + 9x^3$

SOLVING EQUATIONS WITH SQUARES

Solve for x. There may be no solution, one solution, or more than one solution.

13. $x^2 = 25$

14. $36x^2 = 49$

15. $x^2 - 6x + 9 = 0$

16. $x^2 - 6x + 9 = 1$

GRAPHING INEQUALITIES

17. Use graphs to help you find the solution to each of these compound inequalities. In each case, you will need to graph two horizontal lines and one other line.
    a. $-3 < 4x - 3 < 5$
    b. $-3 < -4x + 3 < 5$
    c. $-5 < 4x - 3 < 3$
    d. $-5 < -4x + 3 < 3$

18. Use graphs and tables of values to solve these compound inequalities.
    a. $x - 2 < 3x - 4 < x + 5$
    b. $x - 2 < 3x - 4 < -x + 5$

BILLIONS AND BILLIONS

The following was written on an ice cream package: $3 billion is 1% of the U.S. yearly defense budget. If you ate one ice cream cone per hour per day it would take you 342,466 years to consume 3 billion ice cream cones.

19. Check that the calculation is accurate.

20. Assuming that the information is accurate, what is the U.S. *hourly* defense budget?

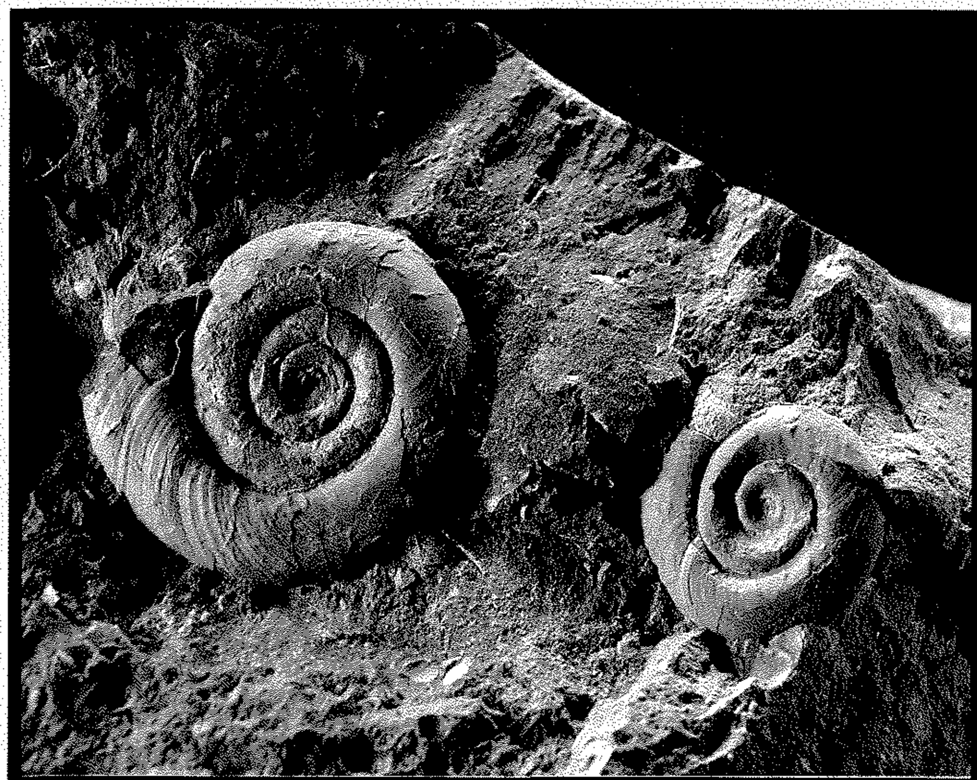
LIGHT-YEARS

21. A *light-year* is the distance light travels in one year. Figure out how far that is in kilometers, given that light travels approximately 299,793 kilometers per second. Use scientific notation.

WHAT A BARGAIN

22. Say that a particularly expensive necklace costs one googol dollars.
    a. Fortunately, it's on sale at 99% off. How much does it cost now?
    b. What percent-off sale would be needed so that the necklace would cost ten billion dollars?

CHAPTER 8



The spiral surface pattern of gastropod fossils

Coming in this chapter:

Exploration A population is growing at a rate of about 2% per year. In how many years will the population double? Experiment with different starting values for the population. How does your answer depend on the starting value?