LESSON 7.9

Powers and Large Numbers

Powers provide a shorthand for writing large numbers. Just as multiplication is repeated addition, raising to a power is repeated multiplication. For example,

$$12 \cdot 12 \cdot 12 \cdot 12 \cdot 12 \cdot 12,$$

which equals 2,985,984, can be written 12^6. Not only is this shorter to write than either the repeated multiplication or the decimal number, it is shorter to key into a scientific calculator.

> **Notation:** Calculators use [^], [x^y], or [y^x] for *exponentiation* (raising to a power). We will use [^] to refer to that key.

Calculators can calculate with exponents that are not positive whole numbers. For example, it is possible to get a value for a number like $3^{-2.4}$ using the key for powers on your calculator. (Try it.) In this lesson, you will consider only *positive whole numbers* for exponents. In later chapters, you will use other exponents.

APPROXIMATING LARGE NUMBERS

1. **Exploration** Consider the number 123,456. Use your calculator to approximate the number as closely as you can with a power of 2, a power of 3, a power of 9, and a power of 10. How close can you get with each power? Repeat this experiment with four other numbers. (Use the same numbers as other students, so as to be able to compare your answers. Use only positive whole numbers having six or more digits.) Is it possible to get close to most large numbers by raising small numbers to a power?

For problems 2-4 find the powers of the following numbers (a-d), that are immediately below and above the given numbers.

a. 2 b. 3

c. 9 d. 10

> **Example:** 691,737 (the population of Virginia, the most heavily populated state in 1790) is:
> a. between 2^{19} and 2^{20}
> b. between 3^{12} and 3^{13}
> c. between 9^6 and 9^7
> d. between 10^5 and 10^6

2. 3,929,214 (the population of the United States in 1790)

3. 48,881,221 (the number of people who voted for President Bush in 1988)

4. 178,098,000 (approximate number of Americans aged 18 or older in 1988)

CLOSER APPROXIMATIONS

It is possible to combine powers with multiplication to get approximations that are closer than those you were able to get in the previous sections by using only powers. For example, the speed of light is approximately 186,282 miles per second. This number is more than 2^{17} and less than 2^{18}, since

$$2^{17} = 131{,}072 \text{ and } 2^{18} = 262{,}144.$$

By multiplying 131,072 by a number less than 2, it is possible to get quite close to 186, 282.

$1.2 \cdot 131{,}072 = 157{,}286.4$ (too small)

$1.5 \cdot 131{,}072 = 196{,}608$ (too large)

$1.4 \cdot 131{,}072 = 183{,}500.8$ (too small, but pretty close)

5. We showed that the speed of light can be roughly approximated by multiplying 2^{17} by 1.4. Find an even better approximation by changing the number by which you multiply, using more places after the decimal point.

You can approximate the speed of light in many different ways using powers of 2. For example:

$93141 \cdot 2^1 = 186{,}282$
$46570 \cdot 2^2 = 186{,}280$
$23280 \cdot 2^3 = 186{,}240$
$45.5 \cdot 2^{12} = 186{,}368$

We used 2^{17} in the example instead of some other power of 2 because it is the *largest* power of 2 that is less than 186,282. We approximated 186,282 by *multiplying that power of 2, by a number between 1 and 2.*

6. Write an approximation to the speed of light using a power of 3 multiplied by a number between 1 and 3. (Hint: Begin by finding the largest power of 3 that is less than 186,282.)

7. Write an approximation to the speed of light using a power of 9 multiplied by a number between 1 and 9.

8. Write an approximation to the speed of light using a power of 10 multiplied by a number between 1 and 10.

9. Combine a power and multiplication to get a close approximation to the length of the Earth's equator, which is 24,902 miles, to the nearest mile. You can use any base for the power, but multiply the power by a number between 1 and the base.

NAMES FOR LARGE NUMBERS

10. Write 100 and 1000 as powers of 10.

There are common names for some of the powers of ten. *Billion* in the U.S. means 10^9, but in Britain it means 10^{12}. The table gives the common names used in the U.S. for some powers of ten.

Power	Name
10^6	million
10^9	billion
10^{12}	trillion
10^{15}	quadrillion
10^{18}	quintillion
10^{21}	sextillion
10^{100}	googol

11. Someone might think a billion is two millions, and a trillion is three millions. In fact, a billion is how many millions? A trillion is how many millions? Explain.

SCIENTIFIC NOTATION

Definition: To write a number in *scientific notation* means to write it as a power of 10 multiplied by a number between 1 and 10. This is the most common way of writing large numbers in science and engineering.

12. Explain why 10 is used for the base in scientific notation rather than some other number. Use examples.

13. Write in scientific notation.
   a. one million
   b. 67 million (the average distance from the sun to Venus in miles)
   c. 5.3 billion (an estimate of the world's population in 1990)
   d. twenty billion
   e. 3.1 trillion (the U.S. national debt in dollars as of June 1990)
   f. three hundred trillion

14. **Project** Find four large numbers that measure some real quantity. They should all be larger than 100,000,000. Encyclopedias, almanacs, and science books are good sources of such numbers.
   a. Tell what each number measures.
   b. Write the number in scientific notation.

REVIEW *PRIME NUMBERS*

There is only one polyomino rectangle of area 2, and only one of area 3. But there are two polyomino rectangles of area 4, corresponding to the products $2 \cdot 2 = 4$ and $1 \cdot 4 = 4$.

15. What is the smallest number that is the area of a polyomino rectangle in 3 different ways? Sketch the three rectangles and show the products.

16. Repeat the problem for 4 different ways.

17. Can you predict the smallest number that is the area of a polyomino rectangle in 5 different ways? Check your prediction.

Definition: Numbers greater than 1 that can only make a rectangle with whole number dimensions in one way are called *prime numbers*.

18. Here is an ancient method, (invented by the Greek mathematician Eratosthenes,) of finding the prime numbers.
   a. On a list of numbers from 1 to 100, cross out the 1.
   Circle 2, cross out its multiples.
   Circle the first number that is not crossed out, cross out its multiples.
   Repeat, until all the numbers are either crossed out or circled.
   b. Explain how and why this method works to find the prime numbers.

19. A mathematician once suggested that *every even number greater than 2 may be the sum of two prime numbers.* No one knows why this should be true, but it has worked for every number that's ever been tried. Test this for yourself with at least ten even numbers. (This is known as *Goldbach's conjecture.* A conjecture is a guess that has not yet been proved true or false.)