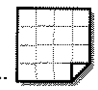
LESSON 14.8

The Golden Ratio

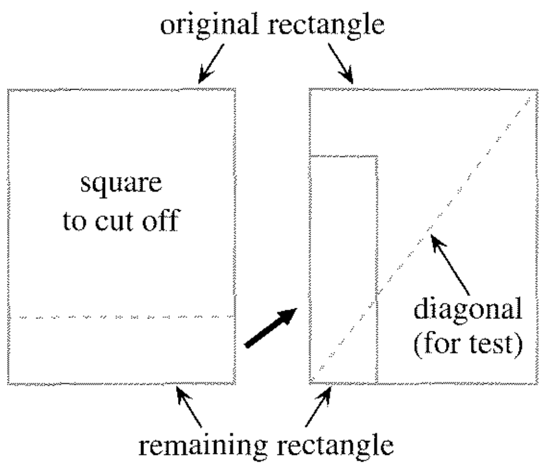
You will need:

scissors 

graph paper 

THE GOLDEN RECTANGLE

1. Take two identical rectangular pieces of paper. Cut a square off one end of one of them, as shown in the figure. Is the remaining rectangle similar to the original one? Check with the diagonal test.



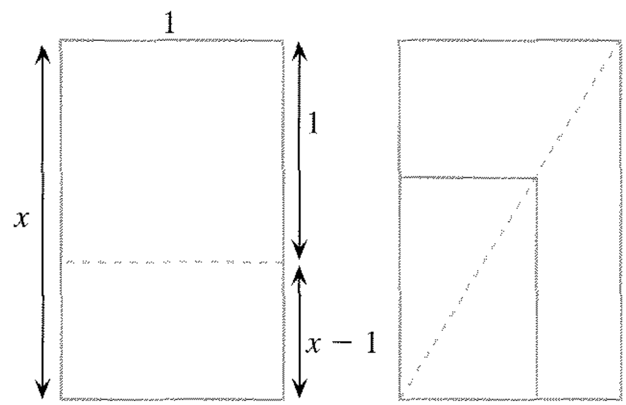
2. **Exploration** Make a paper rectangle, such that the rectangle that remains after cutting off a square is similar to the original rectangle. What are the dimensions of your rectangle? (Hint: Remember that if two rectangles are similar, their length-to-width ratio must be the same. You may use trial and error on your calculators for different sizes, or write and solve equations.)

Definitions:

- A *golden rectangle* is one that satisfies the following property: If you cut a square off one end of the rectangle, the remaining rectangle is similar to the original one.
- The ratio of the longer to the shorter side of a golden rectangle is called the *golden ratio*.

Golden rectangles and the golden ratio are used frequently in art, design, and architecture.

3. What is the length-to-width ratio of the rectangle you found in problem 2? Compare your answer with your classmates' answers.



This figure shows a golden rectangle (on the left). To find the exact value of the golden ratio, we will write and solve an equation about the similar rectangles shown (on the right).

4. Explain why $\frac{x-1}{1} = \frac{1}{x}$.

5. Solve the equation.

There should be two solutions. The positive one is the golden ratio.

6. What is the exact value of the golden ratio?

7. What is the golden ratio, rounded to the nearest one thousandth?

8. What is the reciprocal of the golden ratio, rounded to the nearest one thousandth?

Notation: The golden ratio is often represented by the Greek letter φ (*phi*).

A SPECIAL SEQUENCE

A Fibonacci-like sequence is one in which each term is the sum of the previous two. A geometric sequence is one in which the ratio of consecutive terms is constant. We will try to create a sequence that is geometric and Fibonacci-like at the same time.

9. Consider the sequence $1, k, k^2, k^3, \ldots$. Explain why it is a geometric sequence. What is its common ratio?

10. Explain why, if $1, k, k^2, k^3, \ldots$ were a Fibonacci-like sequence, we would have $1 + k = k^2$.

11. Find a number k that satisfies the equation $1 + k = k^2$. Explain your reasoning.

In problems 9-11, you have shown that the sequence $1, \varphi, \varphi^2, \varphi^3, \ldots$ is geometric and starts out as a Fibonacci-like sequence, since its third term is the sum of the first two. It remains to show that if you add the second and third terms, you get the fourth, if you add the third and fourth, you get the fifth, and so on. More generally, we need to show that if you add the $(n + 1)^{\text{th}}$ term and the $(n + 2)^{\text{th}}$ term, you get the $(n + 3)^{\text{th}}$ term.

12. Use algebra to explain why if $1 + k = k^2$, then $k + k^2 = k^3$.

13. Multiply both sides of $1 + k = k^2$ by k^n. Use the result to show that the sequence $1, \varphi, \varphi^2, \varphi^3, \ldots$ is Fibonacci-like.

GOLDEN WINDOWS

Some architects think that rectangular windows look best if their sides are in the ratio of approximately φ.

14. Imagine that you must make "golden windows" out of square panes. Since the sides must be whole numbers, you will not be able to have an exact golden rectangle, so try to find the dimensions of a few windows having whole number sides in a ratio close to φ.

Many architects use consecutive numbers in the Fibonacci sequence: 1, 1, 2, 3, 5, 8, 13, ... as the dimensions of windows and other rectangles. (Example: 3 by 5, or 5 by 8.)

15. Make a sequence of the ratios of consecutive Fibonacci numbers: 1/1, 2/1, 3/2, 5/3, 8/5, 13/8, ... Are the ratios greater or less than the golden ratio? What is the trend in the long run?

16. a. Plot the points (1, 1), (1, 2), (2, 3), (3, 5), (5, 8), (8, 13), ...
    b. Graph the line $y = \varphi x$.
    c. Describe the position of the points in relation to the line.

17. Plot the points $(1, \varphi), (\varphi, \varphi^2), (\varphi^2, \varphi^3), \ldots$ and the line $y = \varphi x$. Compare the graph with the one in problem 16.

18. **Research** Read about the golden ratio, the golden rectangle, and the Fibonacci sequence. Write a report on what you learn.