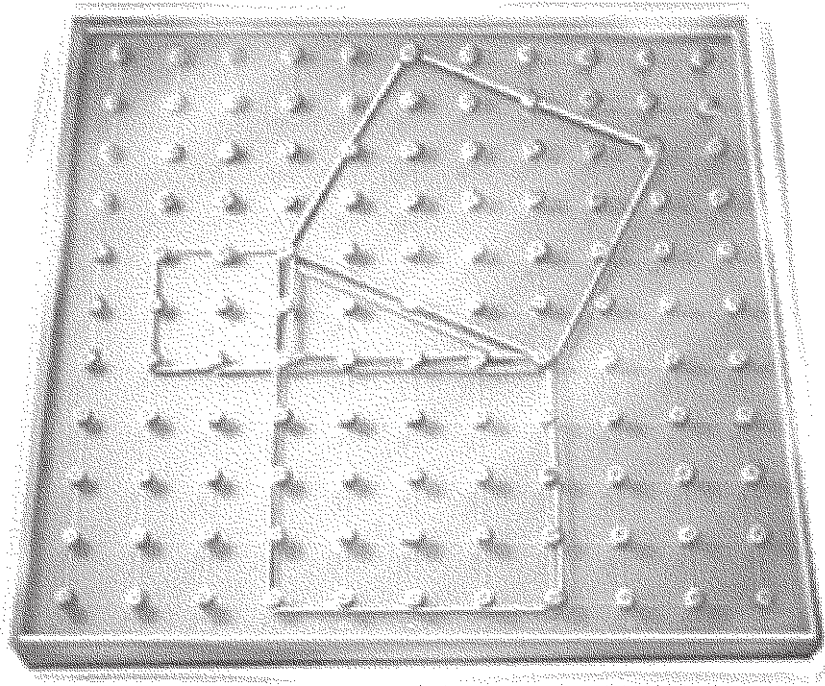# 9 Distance and Square Root

T his chapter, which must be preceded by Lab 8.5 (Geoboard Squares), is built around a hands–on approach to the Pythagorean theorem.

• Mathematically, this approach builds the ideas of square and square root on their geometric interpretation; this improves students' grasp of these operations and of the theorem.

• Pedagogically, since this approach is founded on student–discovered techniques for finding area on the geoboard, it is more likely to be meaningful.

Altogether, this approach provides a powerful complement to the traditional one. In general, I have found it preferable to start with the labs found in this chapter before attempting the traditional geometric proofs of the Pythagorean theorem (although Lab 8.5 provides an excellent preview of those). However, the geoboard work is so different in flavor from the traditional curriculum that either order of presentation may work equally well.

In the course of these lessons, I have found the exploration of taxicab geometry to be useful in a variety of ways.

- First, it helps confront directly the misconceptions that many students already have about distance on a Cartesian grid. By discussing this explicitly, it is easier to clarify what is meant by Euclidean distance.
- Second, studying taxi-circles, taxi-ellipses, taxi-equidistance, and so on helps to throw light on the Euclidean equivalents of these concepts.
- Finally, taxicab geometry provides an example of a non–Euclidean geometry that is accessible to all students and helps convey the creativity and playfulness of pure mathematics.

Of course, it is not necessary to master taxicab geometry beyond the very basics, and this is definitely an "enrichment" topic.

See page 214 for teacher notes to this section.

LAB 9.1
Taxicab Versus Euclidean Distance

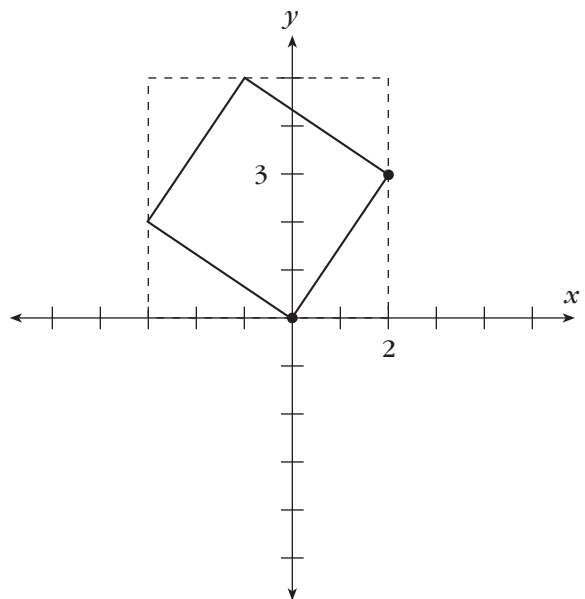Name(s) _____

■ **Equipment:** Geoboard, graph or dot paper

If you can travel only horizontally or vertically (like a taxicab in a city where all streets run North–South and East–West), the distance you have to travel to get from the origin to the point (2, 3) is 5. This is called the taxicab distance between (0, 0) and (2, 3). If, on the other hand, you can go from the origin to (2, 3) in a straight line, the distance you travel is called the Euclidean distance, or just the distance.

Finding taxicab distance: Taxicab distance can be measured between any two points, whether on a street or not. For example, the taxicab distance from (1.2, 3.4) to (9.9, 9.9) is the sum of 8.7 (the horizontal component) and 6.5 (the vertical component), for a total of 15.2.

1. What is the taxicab distance from (2, 3) to the following points?
   a. (7, 9)
   b. (−3, 8)
   c. (2, −1)
   d. (6, 5.4)
   e. (−1.24, 3)
   f. (−1.24, 5.4)

Finding Euclidean distance: There are various ways to calculate Euclidean distance. Here is one method that is based on the sides and areas of squares.

Since the area of the square at right is 13 (why?), the side of the square—and therefore the Euclidean distance from, say, the origin to the point (2,3)—must be $\sqrt{13}$, or approximately 3.606 units.

2. What is the Euclidean distance from (2, 3) to the following points?

   a. (9, 7)

   b. (4, 8)

   c. (2, 5.5)

   d. (6, 0)

   e. (1.1, 3)

3. Find as many geoboard pegs as possible that are at a distance 5 from (5, 5). Record your findings on graph or dot paper.

   a. Using taxicab distance

   b. Using Euclidean distance

Discussion

A. Find a formula for the taxicab distance between two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$. Call the distance $T(P_1, P_2)$. (**Hint:** Start by figuring out a formula for the case where the points are on a common horizontal or vertical line. The formula should work whether P_1 or P_2 is named first.)

B. In Euclidean geometry, for three points A, B, and C, we always have $AB + BC \geq AC$. This is called the *triangle inequality*. Does this work in taxicab geometry? In other words, do we have $T(A, B) + T(B, C) \geq T(A, C)$? If so, in what cases do we have equality?

C. Which is usually greater, taxicab or Euclidean distance? Can they be equal? If so, in what cases?

D. Explain why the answers to Problem 3a are located on what may be called a *taxi-circle*.

LAB 9.2
The Pythagorean Theorem

Name(s) _____

■ **Equipment:** Geoboards, dot paper

1. The figure above shows a right triangle with a square on each side. Find the areas of the squares.

2. Make your own right triangles on geoboards or dot paper, and draw the squares on the sides, as in the figure. Then, working with your neighbors, fill out the table at right. (Note: The "small" and "medium" squares can be the same size.)

3. Describe the pattern of the numbers in the table. It is called the Pythagorean theorem.

4. State the Pythagorean theorem by completing this sentence: "In a right triangle . . ."

Area of squares		
Small	**Medium**	**Large**

Euclidean distance can be calculated between any two points, even if their coordinates are not whole numbers. One method that sometimes works is to draw a slope triangle (i.e., a right triangle with horizontal and vertical legs), using the two points as the endpoints of the hypotenuse. Then use the Pythagorean theorem.

5. What is the Euclidean distance from $(2, 3)$ to the following points?

   a. $(7, 9)$

   b. $(-3, 8)$

   c. $(2, -1)$

   d. $(6, 5.4)$

   e. $(-1.24, 3)$

   f. $(-1.24, 5.4)$

6. Imagine a circle drawn on dot paper with the center on a dot. How many dots does the circle go through if it has the following radii?

   a. 5

   b. 10

   c. $\sqrt{50}$

   d. $\sqrt{65}$

   e. $\sqrt{85}$
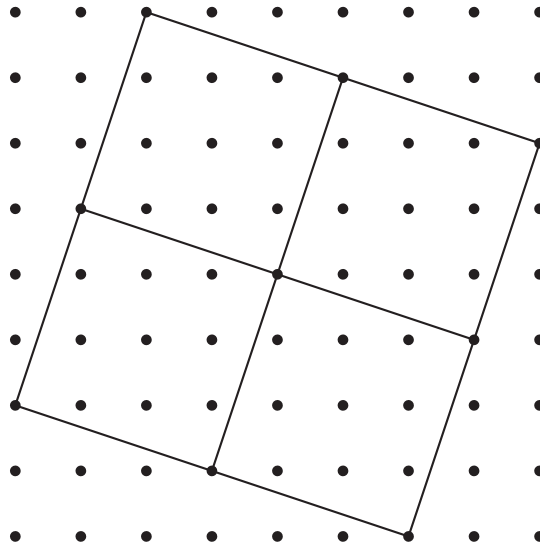
Discussion

A. Repeat Problem 2 with each of the following. Does the Pythagorean theorem work in these cases? If it fails, how?

   a. An acute triangle

   b. An obtuse triangle

B. In what parts of Problem 5 did you *not* use the Pythagorean theorem? How did you find those distances?

C. Find as many geoboard isosceles triangles as possible whose legs share a vertex at the origin and whose bases are *not* horizontal, vertical, or at a 45° angle. (Use Euclidean distance for this puzzle. Limit yourself to examples that can be found on an 11 × 11 geoboard. Problem 6 provides a hint.) Record your findings on graph or dot paper.

LAB 9.3
Simplifying Radicals
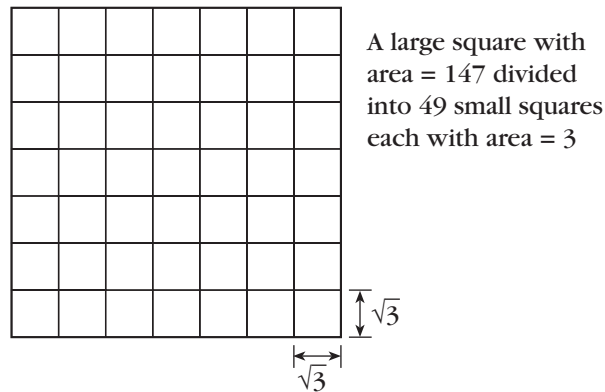
Name(s) _____

■ **Equipment:** Geoboard, dot paper

1. In the above figure, what are the following measures?
   a. The area of one of the small squares
   b. The side of one of the small squares
   c. The area of the large square
   d. The side of the large square

2. Explain, using the answers to Problem 1, why $\sqrt{40} = 2\sqrt{10}$.

3. On the geoboard or dot paper, create a figure to show that $\sqrt{8} = 2\sqrt{2}$, $\sqrt{18} = 3\sqrt{2}$, $\sqrt{32} = 4\sqrt{2}$, and $\sqrt{50} = 5\sqrt{2}$.

4. Repeat Problem 3 for $\sqrt{20} = 2\sqrt{5}$ and so on.

In the figure on the previous page, and in the figures you made in Problems 3 and 4, a larger square is divided up into *a square number of squares*. This is the basic idea for writing square roots in *simple radical form*. The figure need not be made on dot paper. For example, consider $\sqrt{147}$. Since $147 = 3 \cdot 49$, and since 49 is a square number, we can divide a square of area 147 into 49 squares, each of area 3:

A large square with
area = 147 divided
into 49 small squares
each with area = 3

If you pay attention to the sides of the figure, you will see that $\sqrt{147} = 7\sqrt{3}$. Of course, drawing the figure is not necessary.

5. Write the following in simple radical form.

   a. $\sqrt{12}$

   b. $\sqrt{45}$

   c. $\sqrt{24}$

   d. $\sqrt{32}$

   e. $\sqrt{75}$

   f. $\sqrt{98}$

Discussion

A. Draw a figure that illustrates $4\sqrt{5}$ as the square root of a number.

B. Explain how to use a number's greatest square factor to write the square root of that number in simple radical form. Explain how this relates to the figure above.
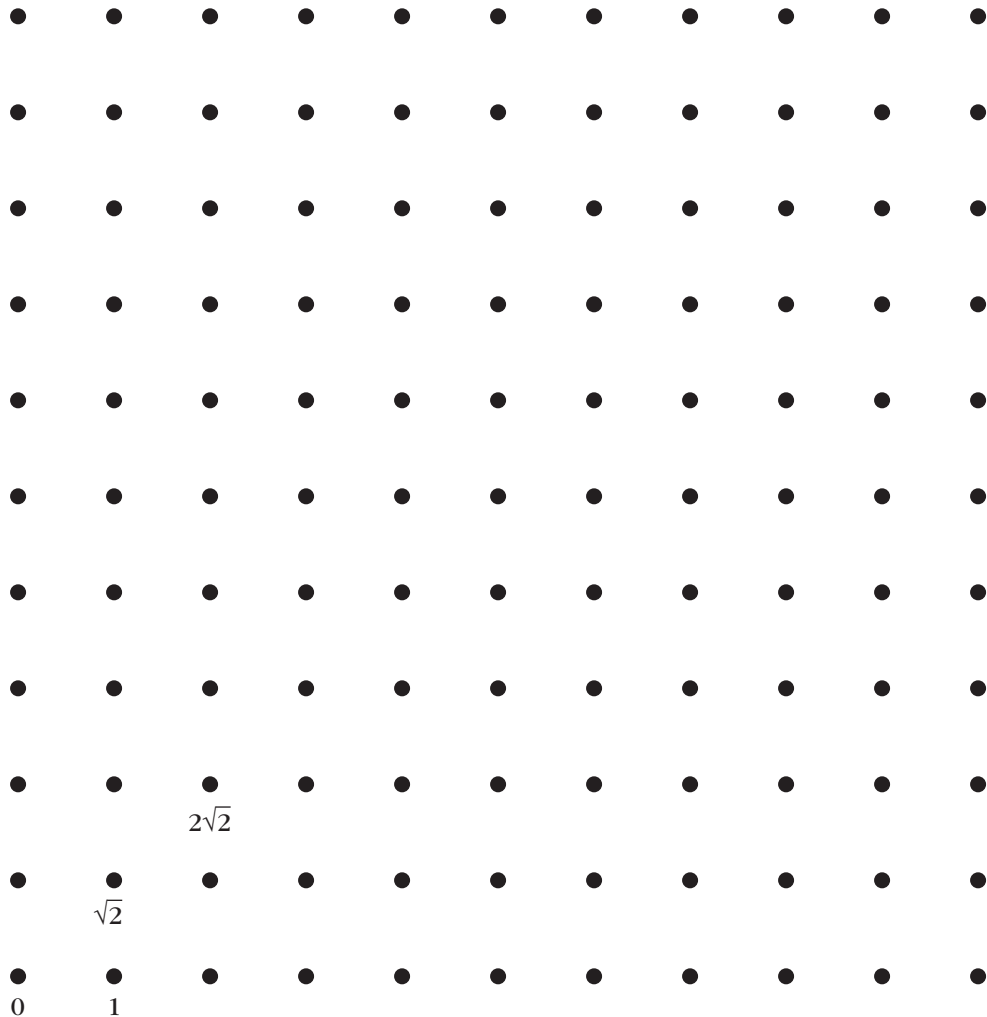
LAB 9.4
Distance from the Origin

■ **Equipment:** Geoboard, dot paper

1. What is the distance from each geoboard peg to the origin? Write your answers in simple radical form on the figure below.

$2\sqrt{2}$

$\sqrt{2}$

0   1

Discussion

A. Discuss any patterns you notice in the distances. Use color to highlight them on the figure. In particular, refer to the following features.
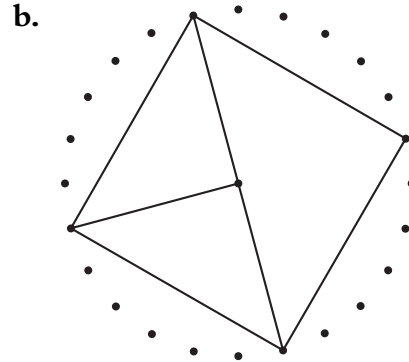
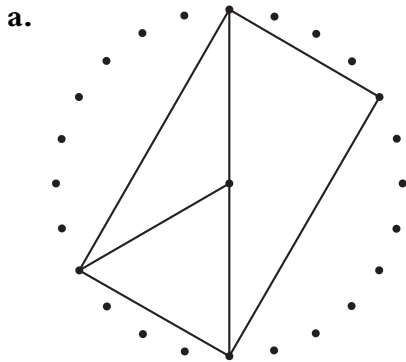 **a.** Symmetry

  b. Slope

LAB 9.5
Area Problems and Puzzles

Name(s) _____

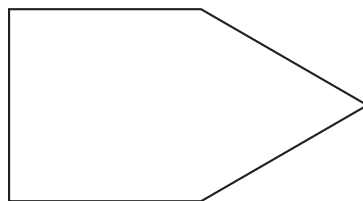■ **Equipment:** Pattern blocks, 1-inch graph paper, assorted dot papers

Problems 1–3 are about the circle geoboard.

a.

b.

1. Find all the angles in the two figures above.

2. Assuming the radius of the circle is 10, find the lengths of all the segments in the two figures.

3. Find the areas of all the triangles in the two figures.

Problems 4 and 5 are about pattern blocks.

4. Find two ways to cover the figure below with pattern blocks.

5. Find the area in square inches of each of the six pattern blocks.

The remaining two problems are substantial and are suitable for long-term projects. The first problem is a puzzle with polyominoes.

6. Draw a pentomino shape (five squares joined edge to edge) on 1-inch graph paper. Figure out how to cut it with as few straight cuts as possible so that the pieces can be reassembled to form a square.

The next problem is about generalizing Pick's formula.

7. Is there a formula similar to Pick's formula for area on a geoboard where the pegs are arranged differently than in a square array? (You may research triangular, hexagonal, and rectangular arrays.)
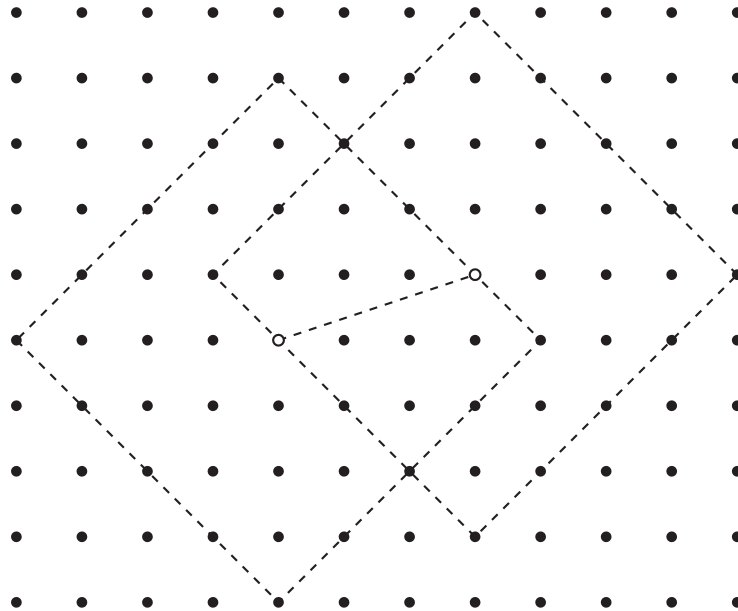
Geometry Labs

■ **Equipment:** Dot or graph paper

These problems are substantial. Do not try to do all of them in a short time!

1. If three points *A, B,* and *C* are plotted so that, in taxicab distances,
   AB = BC = CA, the points are said to form a *taxi-equilateral* triangle.
   Sketch a taxi–equilateral triangle that is

   a. Isosceles in Euclidean geometry (**Hint:** The figure above may help.)

   b. Scalene in Euclidean geometry

Problems 2–5 involve taxicab distances to two points.

2. Sketch the set of points that are equidistant from two points *A* and *B* in the
   following cases.

   a. $A(0, 0)$ and $B(6, 0)$

   b. $A(0, 0)$ and $B(4, 2)$

   c. $A(0, 0)$ and $B(3, 3)$

   d. $A(0, 0)$ and $B(1, 5)$

3. Find the set of points *P* such that $PA + PB = 6$ in the cases listed in Problem 2.

4. Find the set of points *P* such that $PA + PB = 10$ in the cases listed in Problem 2.

5. Find the set of points *P* such that $PA/PB = 2$ in the cases listed in Problem 2.
   (Careful! This last case is particularly tricky.)

Problems 6–9 are about taxi-circles.

6. The number π is the ratio of the perimeter of a circle to its diameter. In Euclidean geometry, $\pi = 3.14159\ldots$. Find the value of taxi-π.

7. Given the line *l* with equation $y = 3x$ and the point $P(3, 5)$, construct a taxi-circle with radius 3 that passes through *P* and touches *l* in exactly one point. (There are two such circles.)

8. Explain how to find the center of a taxi-circle that goes through two points *A* and *B* and, once you have the center, how to sketch the circle. Give examples based on the cases listed in Problem 2.

9. In Euclidean geometry, three noncollinear points determine a unique circle, while three collinear points determine no circle. In taxicab geometry, the situation is somewhat more complicated. Explore different cases, and try to find out when three points determine no circle, one circle, or more than one circle.

Discussion

A. Given two points *A* and *B,* how would you find a third vertex *P* such that $\triangle ABP$ is taxi-isosceles? (**Hint:** There are two cases: $PA = PB$ and $PA = AB$. And, of course, *A* and *B* can be positioned in various ways, such as the ones listed in Problem 2.)

B. Given two points *A* and *B,* find the set of points *P* such that $PA + PB$ is minimal. Investigate the same question for three or even more points.