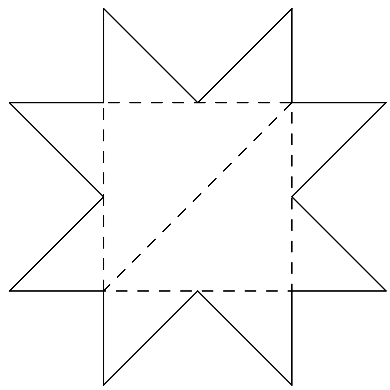
A Comment on Lab 3.9 in *Geometry Labs*

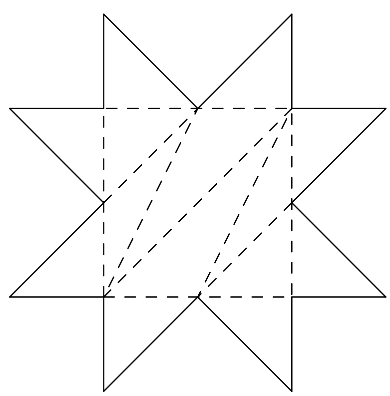
While working on Lab 3.9, "Triangulating Polygons", my 6th and 7th graders initially had a hard time finding a pattern, because some of their "triangulations" had fewer than the expected number of triangles. It turned out that these triangulations violated an unstated assumption, leading to the discoveries outlined in this note.

Here is an example of a flawed triangulation:



We named this example Carter's Abomination, after its creator. With 16 original vertices, 0 inside vertices, and 0 side vertices, this "triangulation" ought to have 14 triangles; instead, it has 10.

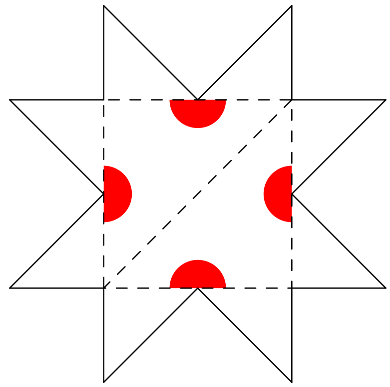
Another student observed that it is possible to continue subdividing Carter's Abomination without introducing new vertices. When thus refined to the furthest possible extent, the triangulation has 14 triangles as expected:



I asked students how to recognize incomplete triangulations and how to account for the "missing" triangles. Here is one response:

> If there is a point in which there is a 180° on it that is not split, there is one less triangle than there should be, because since the point lies on a line, which is a straight angle, we are losing 180°, which is the amount of degrees in a triangle.

In the example above, the "180° angles that are not split" may be observed in the following places:



These angles contribute to the sum of internal angles in the 16-gon, but not to the sum of internal angles in the triangles of the triangulation.

Lab 3.9 asks students to write a formula relating the number of triangles (t) to the number of original vertices (n), the number of inside vertices (i), and the number of side vertices (s). Students may be challenged to revise their formula to also include the number of vertices with the defect described above (d). The formula is then

$$t = n - 2 + 2i + s - d.$$

I think the student explanation of the missing 180° angles hits the nail on the head, but I will also point out another interpretation. In Carter's Abomination, the two large "triangles" may be properly regarded as pentagons, since they each have five vertices (some trios of which happen to be, irrelevantly but deceptively, collinear). The unstated assumption in Lab 3.9 is that each region of a triangulation is a triangle in the *combinatorial* sense—it should have only three vertices. Such triangulations necessarily conform to the expected formula

$$t = n - 2 + 2i + s.$$

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