Chapter 5 Multiplying and Dividing

This chapter focuses on the multiplication and division of expressions that include variables.

New Words and Concepts

The **distributive law** may be the most crucial idea of elementary algebra. By now, especially if they have done the "Make a Rectangle" Explorations, your students should have the intuitive foundation on which to build both understanding of the rule, and proficiency in its application. Note that a Lab Gear version of the law is presented first: "Multiply every block across the top by every block along the side." Then the students are led to discover the usual formulation of the law through their work with the blocks.

The "foil" method for multiplying binomials, while popular with students, is not presented in this book. The reason is that it does not readily generalize to other cases, such as multiplication of a binomial by a trinomial. The formulation: "Multiply every term in the first polynomial by every term in the second polynomial" (which will be stated in Chapter 6) applies with absolute generality to every polynomial multiplication, and is a natural extension of the work with the corner piece.

Cubes and other third degree monomials are introduced, though they will not be used further.

The symbols for **less than or equal to** and **greater than or equal to** are first used in this chapter.

Teaching Tips

The main challenge in this chapter is in combining the "upstairs" method to represent minus with the corner piece. Make sure your students are able to recognize the dimensions of the uncovered rectangle, as most of the work in this chapter depends on it.

Lesson Notes

- **Lesson 1,** Multiplying Monomials, page 56: The ideas presented in this lesson are the foundation of all remaining lessons in this chapter.
- **Lesson 2,** The Uncovered Rectangle, page 57: Though there are probably other ways to show multiplication with minus signs, it is best to follow the format given in this lesson, as it is the standard that will be followed throughout the program.
- **Lesson 3,** Multiplying Polynomials, page 59: Make sure the students cancel matching upstairs and downstairs blocks to get the final answer.
- **Lesson 4,** The Distributive Law, page 60: Problems 9 and 10 force the student to articulate the distributive law.
- **Lesson 5,** More Multiplying, page 61: Make sure the students appreciate that the final, uncovered part of the answer is a rectangle, as it should be, and that it has the right dimensions. Make sure they understand what is meant by the "clockwise" end of the blocks.
- **Lesson 6,** Division and Fractions, page 63: If students have trouble following the examples, start by working out the corresponding multiplication on the overhead projector, then work back.
- **Lesson 7,** More Dividing, page 65: Some trial and error is necessary here.
- **Lesson 8,** Squares and Square Roots, page 67: Students are not asked to find the "square root" of expressions such as $4x^2$, since dealing with the cases of $x > 0$ and $x < 0$ would require more sophistication than beginners usually have. And not dealing with the cases would be quite misleading.
- **Lesson 9,** Three Factors, page 68: The work with three factors is limited by the absence of three-dimensional Lab Gear blocks to represent such quantities as x^3 or x^2y. However, building such blocks out of construction paper should help give the concept more substance. This introduction plants the seed for more work in future years.

Exploration 1 Make a Rectangle

Take blocks for each expression. Arrange the blocks into a rectangle. Write a multiplication equation relating the length, width, and area of the rectangle.

1. $x^2 + 7x$
2. $x^2 + 7x + 6$
3. $x^2 + 7x + 10$
4. $x^2 + 7x + 12$
5. $x^2 + 8x + 12$
6. $x^2 + 13x + 12$

Exploration 2 Make a Square

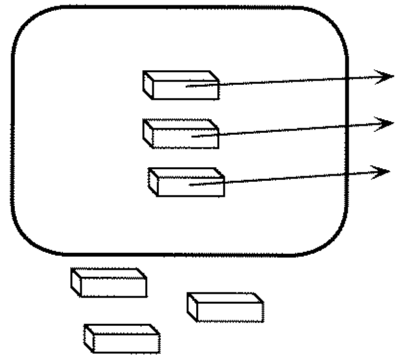
For each problem, arrange the blocks into a square. Not all are possible. Write an equation relating the side length and area of the square.

1. 36
2. 49
3. 40
4. $4x^2$
5. $9x^2$
6. $x^2 + 2x + 1$
7. $x^2 + 6x + 9$
8. $4x^2 + 4x + 1$
9. $x^2 + 8x + 4$
10. $x^2 + 4x + 16$
11. $9x^2 + 12x + 4$
12. $x^2 + 2xy + y^2$

Multiplying Monomials

In some cases, the methods you learned for integer multiplication with the Lab Gear can be used with variables.

For example, to simplify $-3(-x)$, start with zero. Then *take off* $(-x)$ three times. What is left is $3x$, as shown by this figure.



For each expression, write an equivalent expression without parentheses.

1. $(-2)x$
2. $-3(-y^2)$
3. $2(-xy)$
4. Look at problems 1, 2, and 3, and complete the following statements:
   a. When multiplying two monomials that are both preceded by a minus sign, the product ___________.
   b. When multiplying a monomial that is preceded by a minus sign times a monomial that is not, the product ___________.

Exploration 3 Positive or Negative?

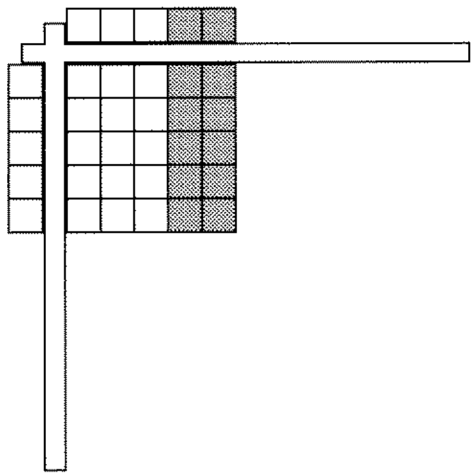
For each expression, write P, N or 0, depending on whether it is positive, negative, or zero. (Try various values for the variables to help you decide. For example, –2, 0, and 2.) Explain your answers.

1. $5x$
2. $-2x^2$
3. $-9y$
4. $5y^2$

The Uncovered Rectangle

This lesson will help prepare you to use the corner piece when minus signs are involved.

For example, this figure shows the multiplication 5(5 – 2).

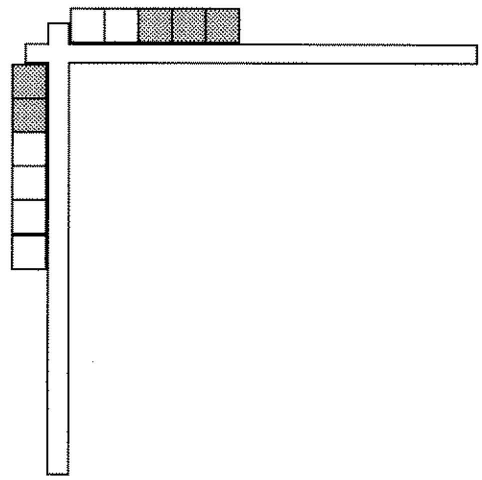


Remember that the shaded blocks are upstairs. Look at the rectangle of downstairs blocks that are not covered by upstairs blocks. We call this the *uncovered rectangle.* The answer to the multiplication is represented by this uncovered rectangle with dimensions 5 and 5 – 2. Of course, the product is 5 times 3, or 15, which is the answer you get when you cancel upstairs and downstairs blocks.

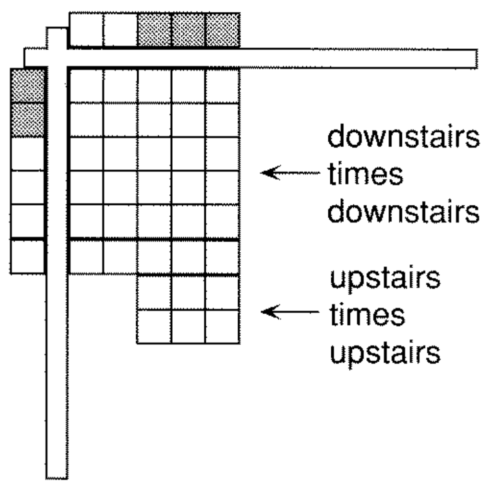
1. Use the corner piece to show the product 3(7 – 2), and sketch your solution.

When there are minus signs in both factors, as in the case of (6 – 2)(5 – 3) you need to set up the problem as shown.

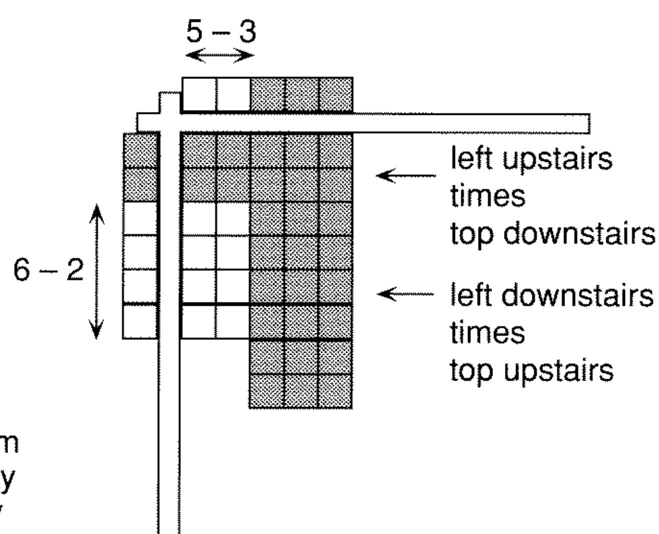
- Put the upstairs blocks at the top end of the left blocks, and the right end of the top blocks. (You can think of it as the *clockwise* end of the blocks.)



- First, multiply the downstairs blocks only.
- Then multiply the upstairs blocks by each other. Since –2(–3) = 6, these blocks must appear downstairs somewhere. You will soon find out why it is convenient to place them at the bottom right as shown on the figure.



- Finally, multiply upstairs blocks on the left with downstairs blocks above, and vice versa, placing them as shown.

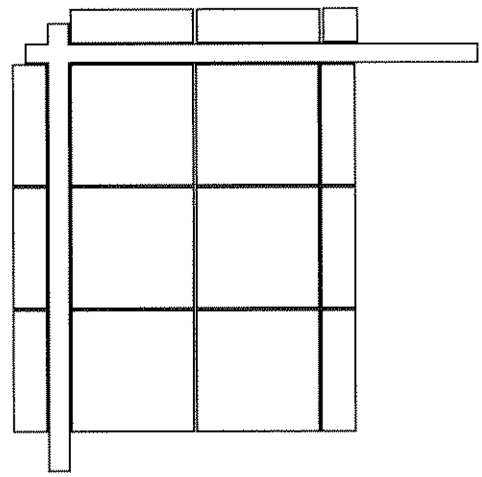


You can now see that the answer (4 times 2 = 8) is shown by the uncovered rectangle.

2. Use the corner piece to show the product (5 – 2)(7 – 4). Sketch your solution.
3. Using arithmetic, the short way to find the answer to the above problem is to multiply 3 times 3. The long way is to multiply every number by every number: 35 – 14 – 20 + 8. Explain how both of these ways can be seen in the blocks.

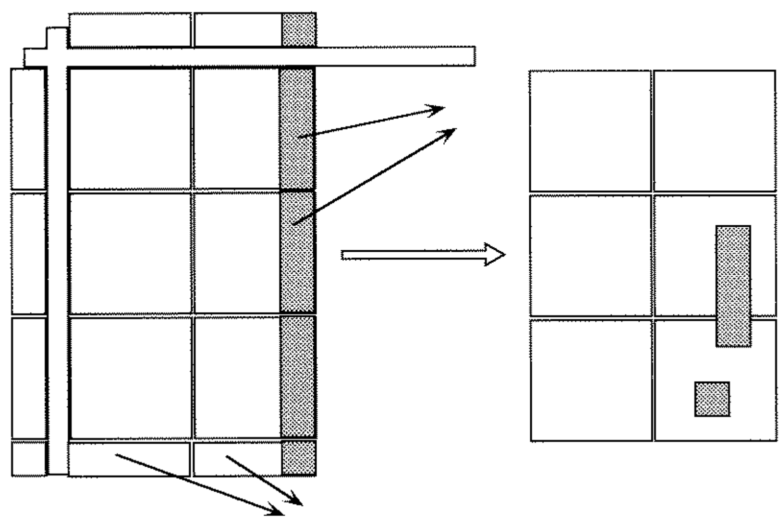
Multiplying Polynomials

This figure will help you remember how to use the corner piece to multiply algebraic expressions. Notice that *every block across the top is multiplied by every block along the side.*



1. Write the multiplication shown by this figure.

This figure shows how to use the corner piece to find the product of polynomials when minus signs are involved. The multiplication is $(3x + 1)(2x - 1)$. Remember that the shaded blocks are *upstairs.*



2. Write the product for this multiplication. Explain what was done to the blocks after using the corner piece.

Notice that, inside the corner piece, the uncovered rectangle has dimensions $3x + 1$ and $2x - 1$. These are the original polynomials we multiplied.

Use the Lab Gear to find these products. Sketch the process as in the above example for problems 3, 6, 9, and 12.

3. $2x(x - 1)$
4. $y(y + 4)$
5. $3x(x + y - 5)$
6. $2y(2x - y + 6)$
7. $3x(2x - 3)$
8. $(x + 5)(3x - 2)$
9. $(y - 4)(y + 3)$
10. $(2x + 4)(x + y + 2)$
11. $x(2y - 5 + x)$
12. $(x - 2)(2x + 3)$
13. $(y + 5)(y + 2x - 3)$
14. $(2y - x - 3)(y + x)$

☑ Self-check

5. $3x^2 + 3xy - 15x$
8. $3x^2 + 13x - 10$
13. $y^2 + 2xy + 2y + 10x - 15$

The Distributive Law

You have investigated ways to remove parentheses from expressions when the parentheses are preceded by a plus or a minus sign. Now look at the question of removing parentheses from expressions involving multiplication.

Use the corner piece to find these products.

1. $x(5 + y)$
2. $(5 + y)x$
3. $y(x + 5)$
4. $(x + 5)y$
5. $5(x + y)$
6. $x(y - 5)$
7. $y(5 - x)$
8. $5(y - x)$
9. Explain how you can correctly remove parentheses from an algebraic expression, when they are preceded or followed by a multiplication.

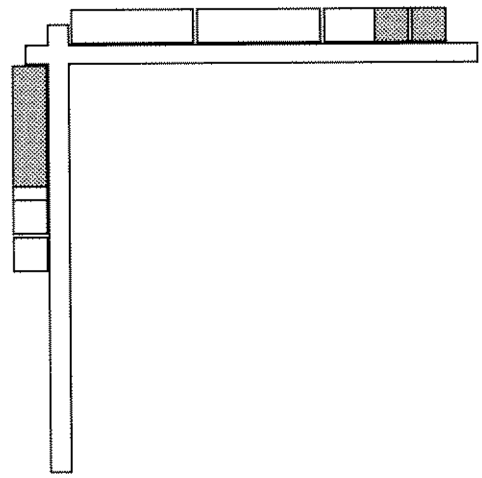
Write equivalent expressions without the parentheses.

10. a. $r(s + t)$
    b. $(r - s)t$
11. a. $-1(x + y)$
    b. $-1(-x + y)$
    c. $-1(x - y)$
12. a. $-(x + y)$
    b. $-(-x + y)$
    c. $-(x - y)$

More Multiplying

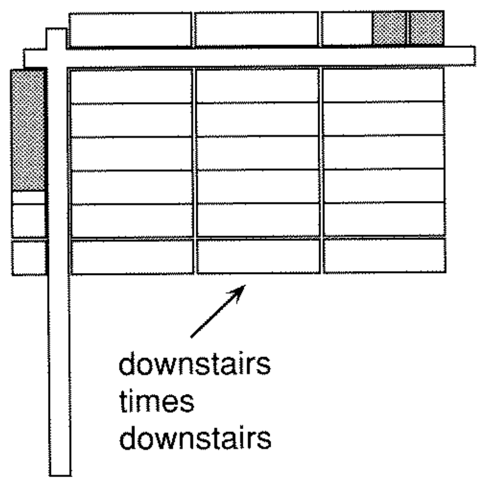
Look at this way to use the Lab Gear to model the multiplication of polynomials when there are minus signs in both factors.

1. Write the polynomials being multiplied.

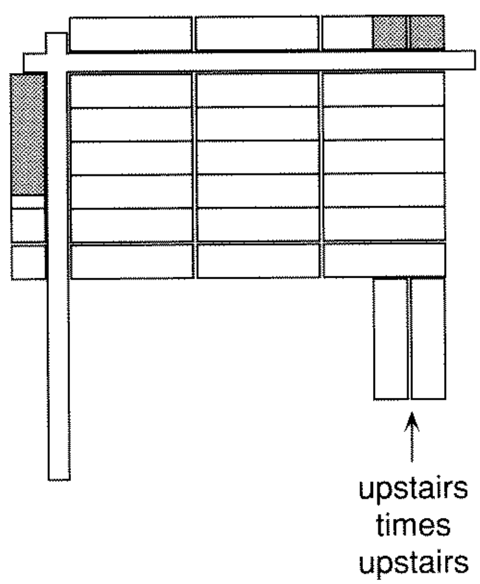


The method you will follow is to multiply all the blocks down the left side by all the blocks across the top. Always be sure to position the upstairs blocks as shown in the figure—they go on the *clockwise* ends of the downstairs blocks.

- First multiply the downstairs blocks only.



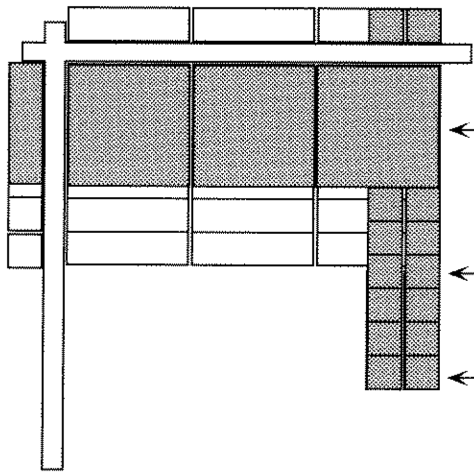
- Then, multiply the upstairs blocks. Since $(-x)(-2) = 2x$, you must put that downstairs somewhere. It is convenient to place the blocks *directly below the −2.*



Now, multiply the upstairs blocks on the left, by the downstairs blocks across the top, and vice versa.

- Multiply left upstairs times top downstairs.
- Now, multiply left downstairs times top upstairs.

2. Write the dimensions of the uncovered rectangle.
3. Write the answer to the multiplication.



For each problem, use the Lab Gear to multiply. Make the uncovered rectangle, then cancel any matching blocks. Write the product.

4. $(4 - x)(2x - 3)$
5. $(y - 5)(2y - 1)$
6. $(3x + 1)(2 + x)$
7. $(6 + y)(2y + 4)$
8. $(3x + 1)(x - 2)$
9. $(y - 4)(2y + 2)$
10. $(2x - 3)(5 - x)$
11. $(y - 3)(y - 5)$
12. $(5 + x)(3x - 3)$
13. $(6 - x)(2x - 3)$
14. $(2y - 2)(6 - y)$
15. $(3x - 1)(2 + x)$

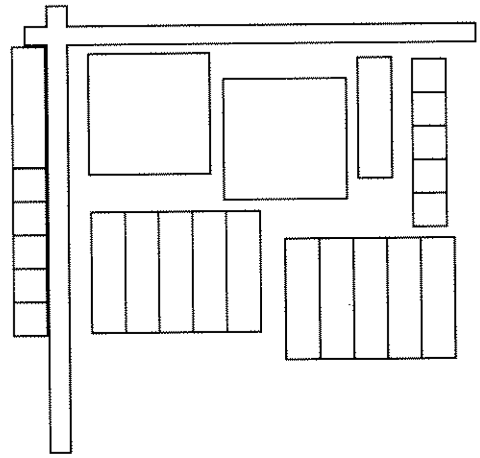
☑ Self-check

4. $-2x^2 + 11x - 12$
10. $-2x^2 + 13x - 15$

Division and Fractions

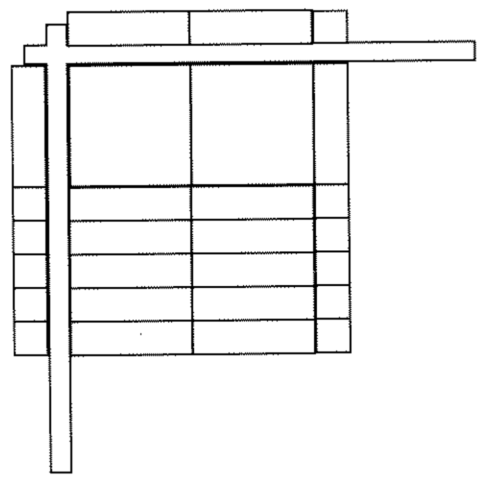
This figure shows the division, $\frac{(2x^2 + 11x + 5)}{(x + 5)}$.

- The numerator is inside the corner piece, the denominator is to the left.



- You arrange the numerator blocks to get a rectangle with one side matching the denominator. The other side is the quotient, and is shown across the top of the corner piece.

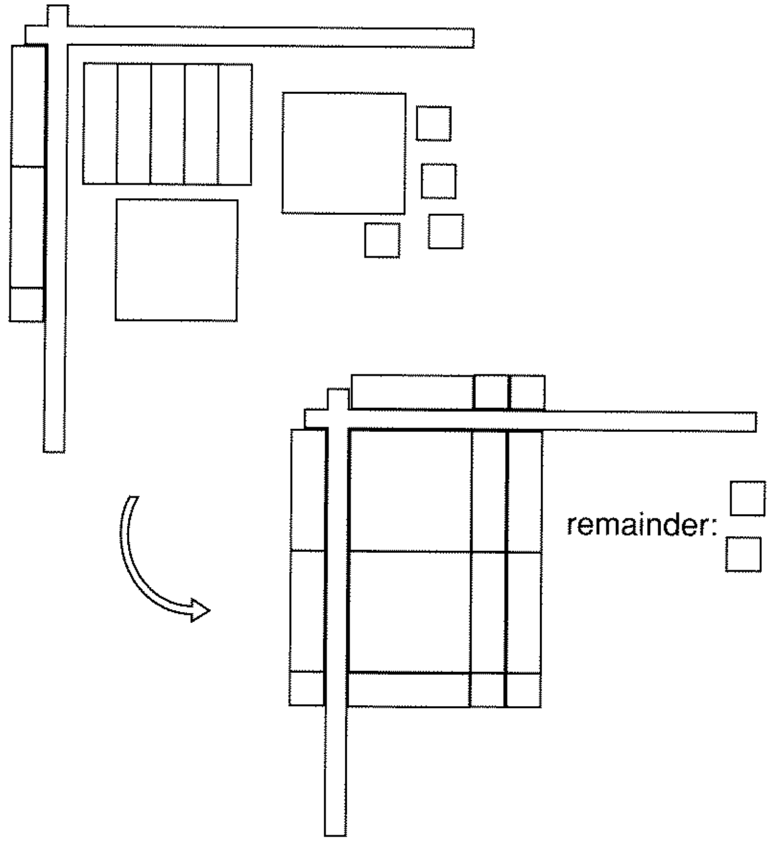
1. Write the quotient.



The denominator was a factor of the numerator, and a rectangle was formed with no pieces left over.

However, in some cases there will be a remainder.

For example, look at these figures.



2. Write the division shown.
3. Write the quotient.
4. Write the remainder.

Use the Lab Gear to show these divisions. Write each quotient and remainder.

5. $\dfrac{6x+4}{2}$
6. $\dfrac{2x^2+4x}{2x}$
7. $\dfrac{9x+3}{3}$
8. $\dfrac{2x^2+6x+4}{x+2}$
9. $\dfrac{3x^2+10x+5}{x+3}$
10. $\dfrac{x^2+3x+8}{x+2}$
11. $\dfrac{4x^2+10x+10}{2x+5}$
12. $\dfrac{2x^2+5x+7}{2x+3}$
13. $\dfrac{2x^2+7x+3}{x+3}$

☑ Self-check

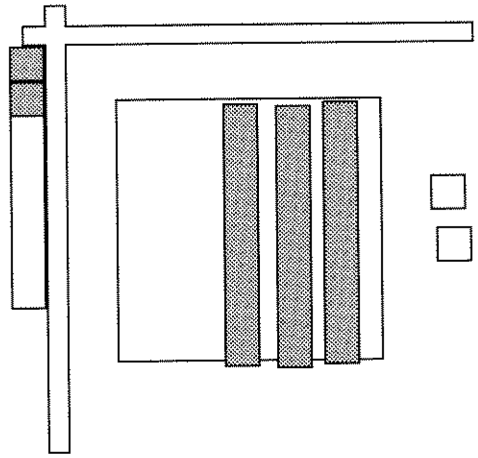
8. $2x + 2$

12. $x + 1$, r 4

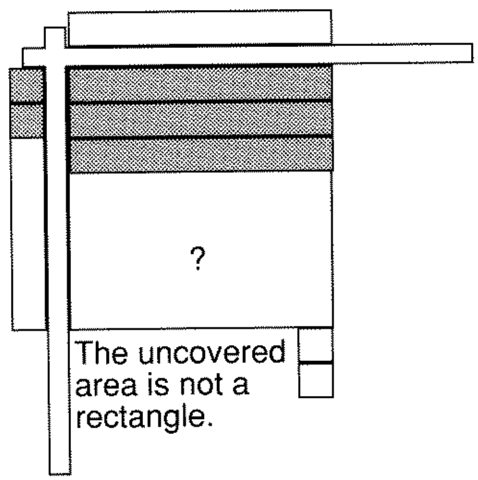
More Dividing

If there is a minus involved, division problems are much harder. Look at this figure.

1. Write the division shown.

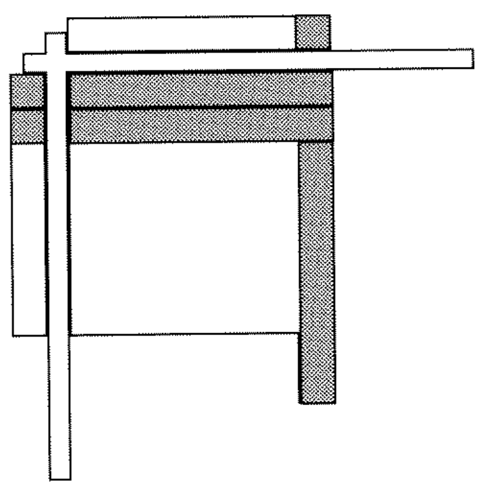


- The arrangement shown in the first try is not correct since the uncovered area is not a rectangle. However, the y^2 and the $-2y$ are placed correctly, because they match the $y - 2$ on the left.



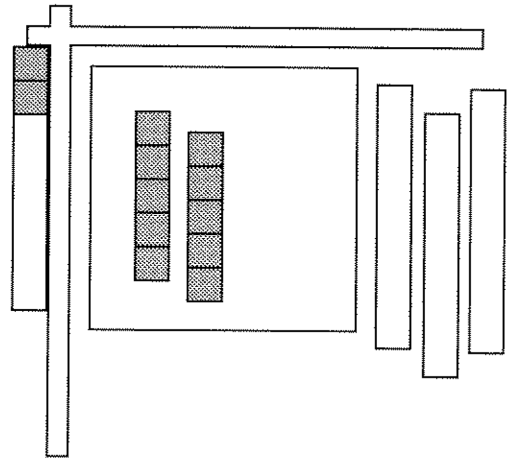
- Moving the other $-y$ does the job. The uncovered area is now a rectangle.

2. Write the quotient.

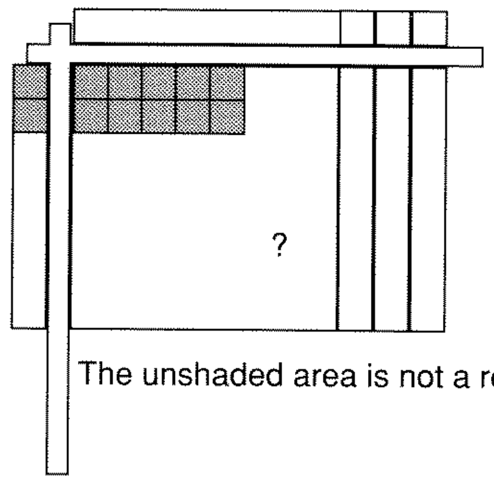


The next example shows once again that adding zero is a useful technique.

3. Write the division shown.



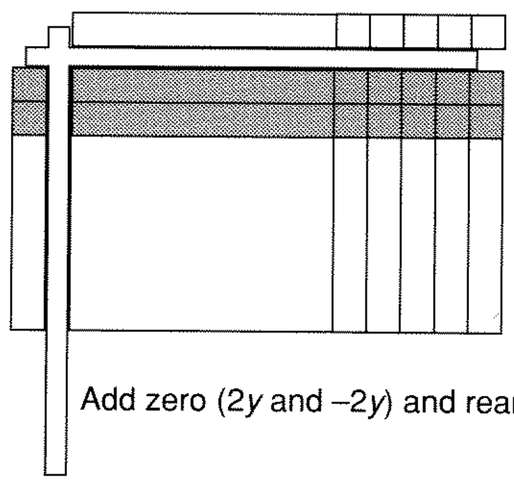
Here is an attempt at arranging the quantity inside the corner piece into a rectangle. This did not work.



The unshaded area is not a rectangle.

However, after adding zero in the form of $2y$ upstairs and $2y$ downstairs, it is possible to get an uncovered rectangle.

4. Write the quotient.



Add zero ($2y$ and $-2y$) and rearrange.

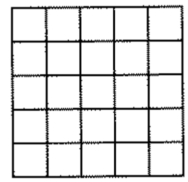
Try these division problems with the Lab Gear.

5. $\frac{y^2 + y - 6}{y + 3}$
6. $\frac{3x^2 - 9x + 6}{x - 2}$
7. $\frac{2x^2 + 7x + 3}{2x + 1}$
8. $\frac{y^2 - 6y + 8}{y - 4}$
9. $\frac{x^2 + 5x + 1}{x - 1}$
10. $\frac{3x^2 + 13x + 4}{x + 4}$
11. $\frac{x^2 - x - 6}{x + 2}$
12. $\frac{2y^2 + y + 5}{y - 2}$

Squares and Square Roots

The *square* of a positive number can be shown by the area of a square having that number as a side. The positive *square root* of a number can be shown by the side of a square having that number as its area.

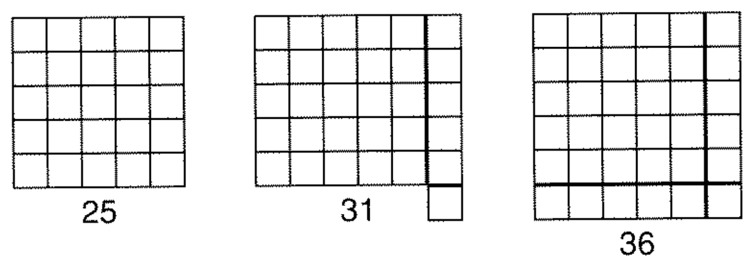
- For example, to find the square of 5, make a square with side 5, and find its area. The answer is 25. You write $5^2 = 25$.
- To find the positive square root of 25, make a square with area 25, and find its side. The answer is 5. You write $\sqrt{25} = 5$.



The square root symbol is called a *radical.*

1. Find the square of 9.
2. Find the square root of 9.
3. Find the square root of 36.

If a number cannot be made into a square with the blocks, its square root is not a whole number. For example, 31 is greater than 25, and smaller than 36, and cannot be shown by a square. Its square root is greater than 5 and less than 6. A calculator provides a more exact value, 5.567764363.



For each number below, the positive square root is between two consecutive whole numbers. Write those two numbers.

4. 42
5. 87
6. 52
7. 114

Find the squares of these expressions. Use the corner piece.

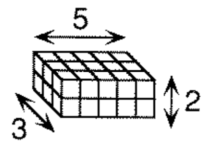
8. $2x$
9. $2 + x$
10. $3y$
11. $3 + y$

☑ Self-check

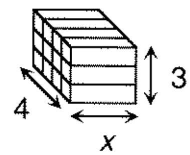
8. $4x^2$

Three Factors

The multiplication equation $3 \cdot 5 \cdot 2 = 30$ can be shown with the Lab Gear like this.



The multiplication expression $3 \cdot 4 \cdot x$ looks like this.



Build these multiplication expressions with your blocks.

1. $5 \cdot 2 \cdot y$
2. $4 \cdot x \cdot 2$
3. $3 \cdot y \cdot y$
4. $8 \cdot 1 \cdot x$
5. $x \cdot y \cdot 5$
6. $x \cdot 2 \cdot x$

Using heavy paper, make boxes to represent each of these expressions.

7. $x^2y = x \cdot x \cdot y$
8. $xy^2 = x \cdot y \cdot y$
9. $x^3 = x \cdot x \cdot x$
10. $y^3 = y \cdot y \cdot y$

When a number is multiplied by itself three times, as in problems 9 and 10, it is called **cubing** the number. For example, 8 is the cube of 2, since $8 = 2 \cdot 2 \cdot 2 = 2^3$.

Notice that when you are working with three factors, upstairs no longer means minus!

11. Why do you think x^3 is called the cube of x?
12. What do you think "cube root" means? What number would be the cube root of 27?

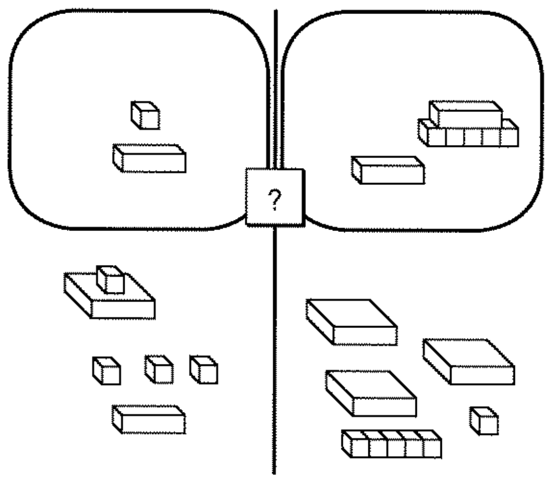
Exploration 4 Which is Greater?

For each of these problems, compare the two expressions by trying different values for x. Good values to try are –2, –1, $-\frac{1}{2}$, 0, $\frac{1}{2}$, 1, and 2. Rewrite each sentence, substituting the correct symbol (<, ≤, =, ≥, >) for the question mark. (≤ means **less than or equal to**, and ≥ means **greater than or equal to**.) Explain your thinking if you say it is impossible to decide which is greater.

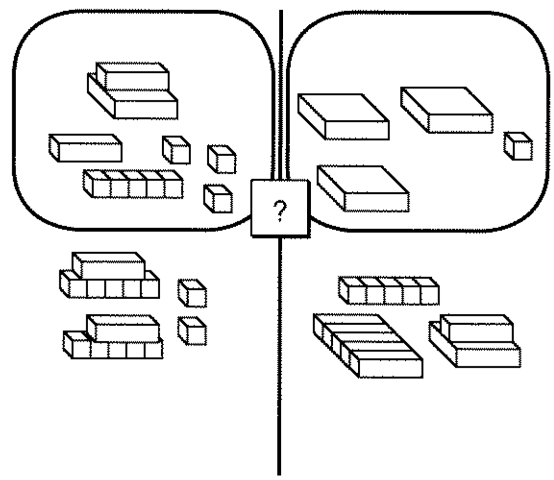
1. x **?** $2x$
2. x^2 **?** x
3. x^2 **?** -1
4. x^2 **?** 0

For these problems, simplify both sides first. Use the Lab Gear.

5.



6.



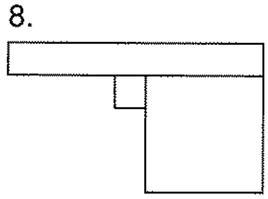
Exploration 5 Perimeter

Use an xy-block and a 5-block to make shapes with these perimeters. Sketch the shape in each case.

1. $2x + 2y + 2$
2. $2x + 2y + 10$
3. $2y + 12$
4. Repeat problems 1–3, using a y-block and a $5x$-block.
5. Use another combination of blocks to get a perimeter of $2x + 2y + 2$.
6. Use another combination of blocks to get a perimeter of $2x + 2y + 10$.
7. Use another combination of blocks to get a perimeter of $2y + 12$.

Write the perimeter of each of these figures.

8.



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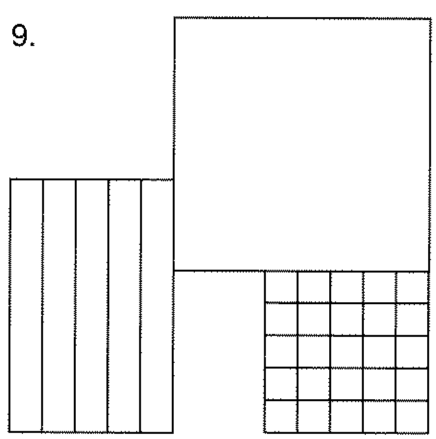

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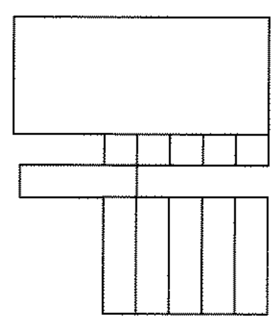
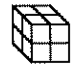
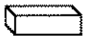
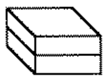
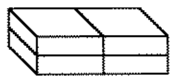
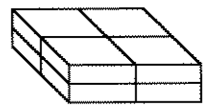


Exploration 6 Volume and Surface Area

Write the volume and surface area for each of these Lab Gear buildings.

1. 
2. 
3. 
4. 
5. 
6. 
7. 
8. 
9. 
10. 
11. 
12. 

Describe and sketch the next building in these sequences. Compare your answers with your classmates'. For each, find the volume and surface area.

13. Buildings 1-4.
14. Buildings 5-8.
15. Buildings 9-12.

Exploration 7 Always, Sometimes, or Never?

Write A, S, or N, depending on whether the statement is always, sometimes, or never true. Explain your answers.

1. $3(2y - 5) = 3 \cdot 2y - 5$
2. $4x(1 + 7) = 4x \cdot 1 + 7$
3. $-3 - (y^2 + 2) = -3 - y^2 - 2$
4. $2x(x + 5) = 2x^2 + 10x$
5. $10 + (3y + 9) = 10 + 3y + 9$
6. $25 - (5 - x) = 25 - 5 - x$