Functions

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Investigating Rectangle Areas

In this lesson, we will investigate rectangle areas by looking for patterns in tables and graphs. These are the questions we will be thinking about:

◊ How does the area of a rectangle change if you vary the length or width and leave the other dimension unchanged?

◊ How does the area of a rectangle change if you vary both the length and the width?

1. What is the area of a rectangle with the following dimensions?
   a. 1 by 9
   b. 2 by 9
   c. 3 by 9
   d. 9 by 9

2. What is the area of a rectangle with the following dimensions if x = 10?
   a. 1 by x
   b. 2 by x
   c. 3 by x
   d. x by x

3. Fill out this table:

Rectangle area	
x	1 by x
1	1
2	
3	
4	
5	
6	

4. Draw axes, with x on the horizontal axis, and area on the vertical axis. Plot the points you obtained in the previous exercise for the area of 1 by x rectangles. For example, (1,1) will be on the graph.

5. Does it make sense to connect the points you plotted? What would be the meaning of points on the line, in between the ones you got from your table?

6. Now fill out these tables:

Rectangle area	
x	2 by x
1	
2	4
3	
4	
5	
6	

Rectangle area	
x	3 by x
1	
2	
3	9
4	
5	
6	

Square area	
x	x by x
1	
2	
3	
4	16
5	
6	

7. On the same axes, graph the data you obtained for 2 by x, 3 by x, and x by x rectangles. For more accuracy on the last one, you may use your calculator to find points for x = .5, 1.5, and so on.

8. **Discussion:** Answer these questions about the graphs:
   a. Which ones represent a proportional relationship?
   b. How do the first three graphs differ from each other? Explain.
   c. What is special about the fourth graph?
   d. Do the graphs intersect? What is the meaning of the intersections?
   e. Where would the graphs meet the y-axis if we extended them?
   f. Which area grows the fastest? why?

9. I am thinking of a number, which I will call k. If I made a table and a graph for rectangles with dimensions k by x, it would start like this:

Rectangle area	
x	k by x
1	5

   What is k equal to? Explain.

10. Here is one part of a table:

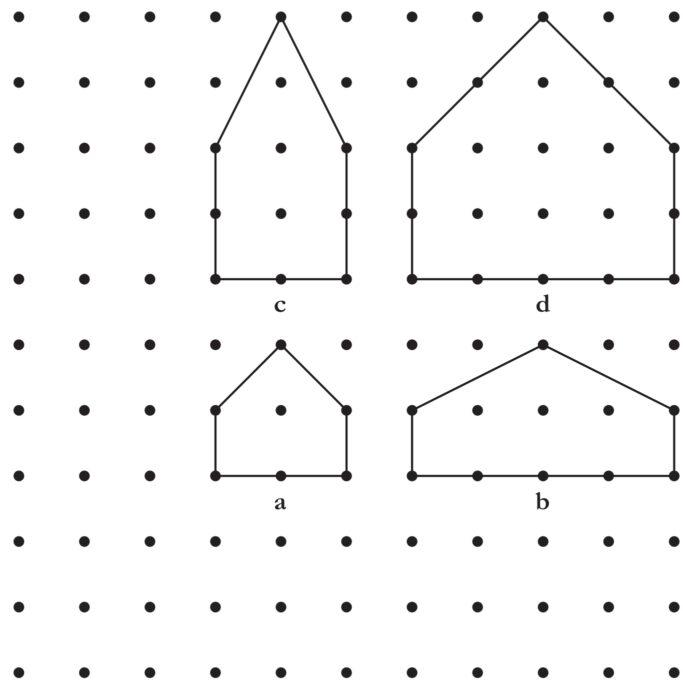
Rectangle area	
x	9 by x
?	13.5

   What is the missing number? Explain.

LAB 10.1
Scaling on the Geoboard

Name(s) ______________________

■ **Equipment:** 11 × 11 geoboard, dot paper



The figure shows four houses. House **a** is the original, and the others have been copied from it following very exact rules.

1. What is the rule that was used for each copy?

2. Among the three copies, two are distorted, and one is a scaled image of the original.

a. Which one is scaled?

b. How would you describe the distortions?

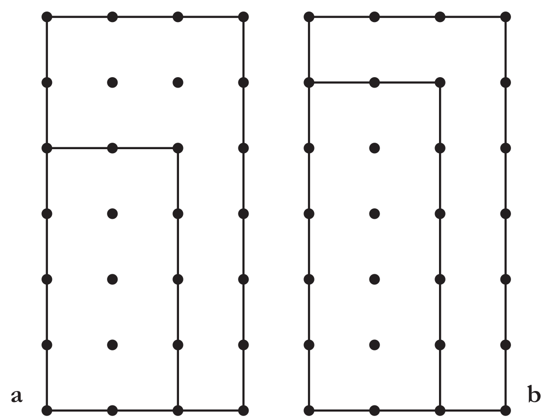
When two figures are scaled images of each other, they are said to be *similar*. Similar figures have *equal angles* and *proportional sides*. The sides of one figure can be obtained by multiplying the sides of the other by one number called the *scaling factor*.

3. What is the scaling factor that relates the two similar houses in the figure above?

LAB 10.2
Similar Rectangles

Name(s) ______________________

■ **Equipment:** 11 × 11 geoboard



The figure above shows geoboard rectangles nested inside each other.

1. Explain why the two rectangles on the left are similar but the two rectangles on the right are not.

2. Draw the diagonal from the bottom left to the top right of the larger rectangle in **a**. Note that it goes through the top right vertex of the smaller rectangle. Repeat on **b**. How is it different?

3. The rectangles in **a** are part of a family of similar geoboard rectangles. If we include only rectangles with a vertex at the origin (the bottom left peg of the geoboard), the family includes ten rectangles (five horizontal ones and five vertical ones—two of the vertical ones are shown in the figure).

a. List the rectangles in that family by listing the coordinates of their top right vertex.

b. What is the slope of the diagonal through the origin for the vertical rectangles?

c. What is the slope of the diagonal for the horizontal rectangles?

4. Including the one in Problem 3, there are ten families of similar rectangles (including squares) on the geoboard. Working with your neighbors, find them all, and list all the rectangles in each family. (Again, assume a vertex at the origin, and use the coordinates of the top right vertex for the list. Only consider families with more than one rectangle.)

5. Working with your neighbors, find every geoboard slope between 1 and 2. (Express the slopes both as fractions and as decimals.) Counting 1 and 2, there are seventeen different slopes.

Discussion

A. Problem 2 is an example of using the *diagonal test for similar rectangles*. Explain.

B. In Problems 3 and 4, how does symmetry facilitate the job of listing the rectangles?

C. What are the advantages of fractional versus decimal notation in Problem 5?

D. Explain how to use the answers to Problem 5 to create lists of geoboard slopes in the following ranges.

  a. Between 0.5 and 1

  b. Between –1 and –2

  c. Between –0.5 and –1

Guess My Function

Here is an example of a function: $y = 2x - 3$.

For this function,
If $x = 0$, $y = -3$.
If $x = 1$, $y = -1$.

1. a. If $x = 2$, $y = ?$
   b. If $x = 1.5$, $y = ?$
   c. If $x = -1$, $y = ?$

x is called the *independent variable*, or input. For this function, you can choose any number for x.

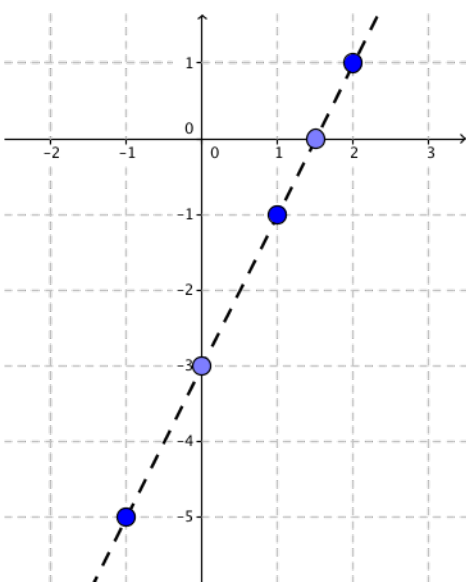
y depends on x, so it is called the *dependent variable*, or output.

You can arrange the information about this function in a table:

x	y
-1	
0	-3
1	-1
1.5	
2	

2. Fill out the rest of the table.

You can also arrange the information in a graph:



3. Label the points with their coordinates.

Definition: A *function* is a rule that assigns to each input exactly one output.

On this page, we made a table and a graph from knowing the formula for the function.

On the next page, you will guess the formulas for functions, knowing a table or a graph.

4. Guess the formula of the function for each table. (Hint: what can you do to x to get y?)

a.

x	**y**
-2	0
-1	1
0	2
1	3
2	4

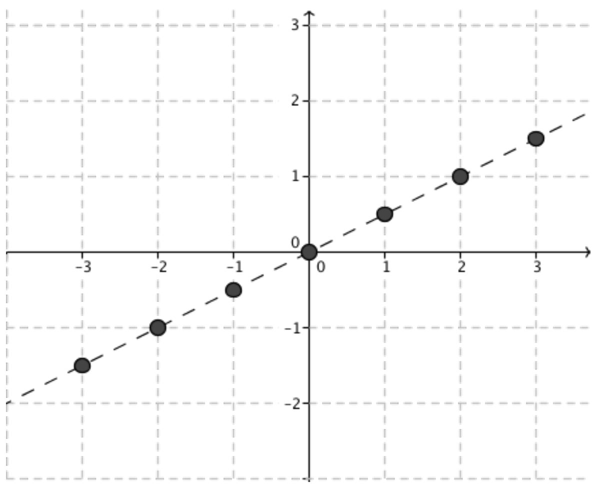
b.

x	**y**
-2	-6
-1	-3
0	0
1	3
2	6

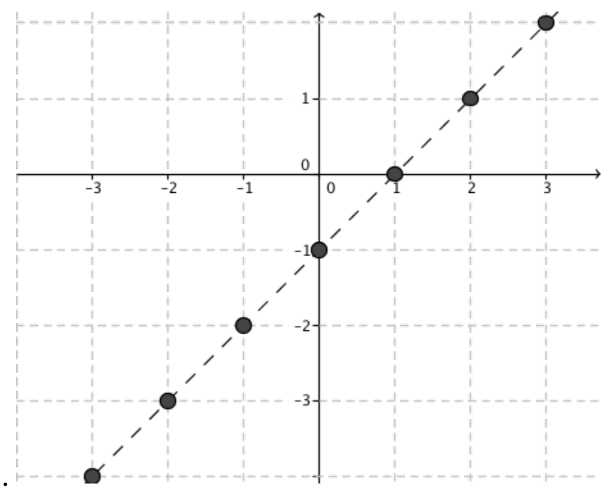
c.

x	**y**
-2	-6
-1	-5
0	-4
1	-3
2	-2

5. Guess the formula of the function for each graph. (Hint: label the points with their coordinates.)



a.



b.

6. Guess the formula of the function for each table. These are more challenging.

a.

x	**y**
-2	6
-1	5
0	4
1	3

b.

x	**y**
-2	-7
-1	-5
0	-3
1	-1

c.

x	**y**
-2	3
-1	0
0	-1
1	0

Functions appear in all sorts of situation in math. For example, the input could be the side of a square, and the output its area. In that case, the formula would be $A=s^2$.

7. For these functions, write a formula, and name the input and the output:
   a. The perimeter of a square
   b. Half of a number
   c. The area of a circle

Discounts

A discount card at a movie theater costs \$10. With that card, it only costs \$3 to attend a movie, instead of \$5. The card is valid for three months.

1. Use the same pair of axes for both of the graphs in this problem. Make a graph of the *total cost* (including the cost of the discount card if you got one) as a function of the *number of movies* you see:
   a. if you have the discount card.
   b. if you do not have the discount card.

2. What is the total cost of seeing n movies in three months
   a. with the discount card?
   b. without the discount card?

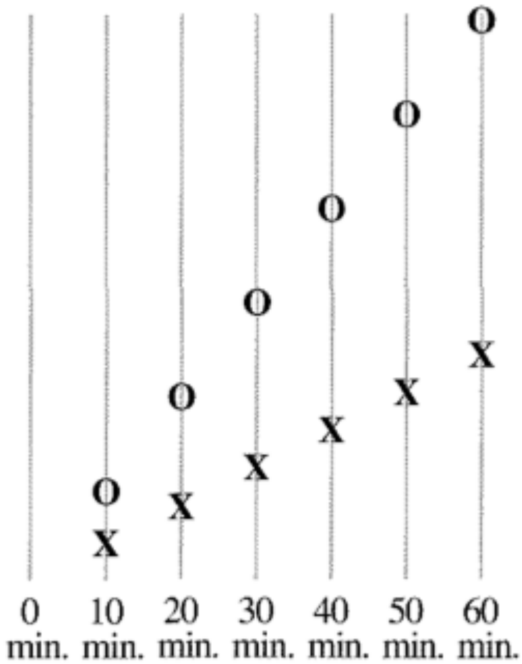
3. a. Which of those two choices is a proportional relationship?
   b. **Bonus:** Can you think of another situation involving money that is a proportional relationship?

4. a. If you saw 12 movies in three months, how much would you save by buying the discount card?
   b. If you saw only 2 movies in three months, how much would you save by *not* buying the discount card?

5. What is the break-even point; that is, how many movies would you have to see in order to spend exactly the same amount with and without the discount card?

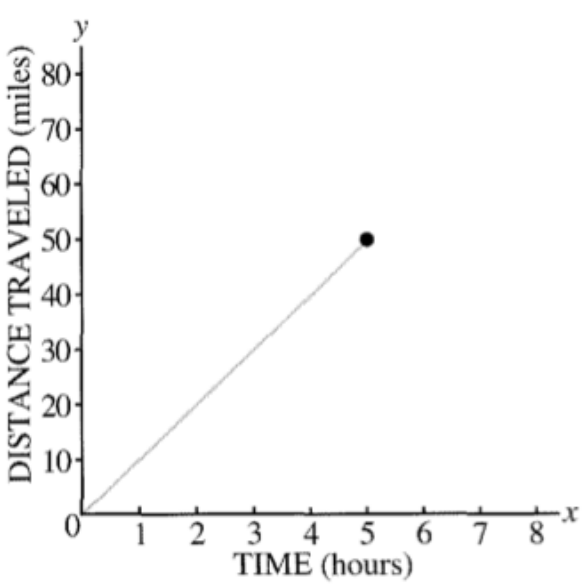
6. How would your decision be affected if the cost of the discount card were raised to \$12?

Constant Speed



1. Ophelia and Xavier are traveling along a road. If you could view the road from above and make a sketch of what you saw every ten minutes, your sketches might look something like the figure.
   a. Which person (0 or X) is traveling faster?
   b. If the entire length of the road is six miles, can you figure out approximately how fast each person is traveling? Explain.

Bea is participating in a long-distance roller-skating race. Her speed is approximately 10 miles per hour. The graph below shows Bea's progress. It shows that after 5 hours she had traveled 50 miles.



2. Bea's motion:
   a. Copy the graph onto graph paper. Use a whole piece of graph paper. You will be adding more to this graph.
   b. One of the points on the graph is (5,50). Mark and label three more points on the graph of Bea's progress.

3. The distance traveled and the time elapsed are in a proportional relationship. Explain how we know this.

4. In this lesson we are assuming everyone travels at a constant speed.
   a. What might make it impossible to travel at a constant speed? Explain.
   b. Why might it be a good idea to assume constant speed anyway?

Abe is walking along the same road. Assume he is moving at an approximately constant speed. The table shows how long it took for Abe to go certain distances.

Abe's Progress

Time (hours)	Distance (miles)
1	4
2	8

5. a. Copy and complete the table up to 20 miles.
   b. Use the same axes you used for Bea. Plot and label the points from the table in part (a).
   c. Connect the points with a straight line. Then find and label a point that is on the line but not in your table. Interpret the coordinates of the point in terms of this problem.

Amazingly, Gabe and Al are on the same road, and started at the same time. Gabe is riding a scooter, going 30 mph. Al is driving a van, going 50 mph.

6. Make tables like the one you made for Abe showing Gabe's progress on his scooter and Al's progress in the van. Make graphs of their progress on the same axes you used to show Abe's and Bea's progress. Label the four different lines.

7. Use your graphs to help you answer these questions. If Bea and Abe start out at the same time,
   a. how far apart will they be after one hour?
   b. how far apart will they be after two hours?

8. Look for a pattern.
   a. How far apart will Abe and Bea be after H hours? Explain.
   b. Is this a proportional relationship? Explain.

9. Mrs. Gral was traveling at a constant speed. She started on the same road at the same time as all the others, and was two miles ahead of Abe after one hour.
   a. Add a graph of Mrs. Gral's progress to your axes.
   b. How far ahead was Mrs. Gral after two hours?
   c. Was she ahead of, or behind Bea? After three hours, how far ahead or behind?
   d. How fast was Mrs. Gral going? What mode of travel do you think she was using?

10. **Summary:**
   a. How does the mode of travel affect the steepness of the line? Explain.
   b. What is the meaning of points on two of the graphs that have the same x-coordinate but different y-coordinates?
   c. What is the meaning of the vertical distance between two lines for a given value of x?

In the Lab

A Mystery Liquid

Reg, Bea and Gabe were doing an experiment in science class. They had an unknown liquid whose volume they measured in a graduated cylinder. A graduated cylinder is a tall, narrow container that is used for accurately measuring liquid volume. They used one that weighed 50 grams, and measured volume in milliliters. They used a balance to find the mass of the liquid to the nearest gram.

Reg's data

volume	mass
10 ml	16 g
20 ml	32 g
50 ml	80 g
80 ml	128 g

1. a. Plot Reg's data, with *volume* on the horizontal axis and *mass* on the vertical axis.
   b. Does it make sense to connect the points on your graph? Explain.

2. Find an equation relating mass to volume.

3. Estimate the mass of
   a. 60 ml of liquid
   b. 1 ml of liquid

4. If you add 30 ml to the volume, how much are you adding to the mass? See if you get the same answer in two different cases.

5. If you double the volume, do you double the mass?

Bea's data

mass	volume
16 g	10 ml
32 g	20 ml
48 g	30 ml
64 g	40 ml

6. Plot Bea's data on the same axes, using a different symbol or color for those points. Be careful! Because the mass depends on the volume, make sure the volume is on the horizontal axis, and the mass is on the vertical axis, again.

7. Connect the points on your graph with a line and write an equation for the line.

8. Estimate the volume of:
   a. 100 g of liquid
   b. 1 g of liquid

9. Compare Bea's graph with Reg's graph. Explain the similarities and differences.

10. If you add 10 ml to the volume, how much are you adding to the mass? See if you get the same answer in three different cases. Is the answer consistent with what you found in Reg's data?

Definition: *Density* equals mass per unit of volume. This means that to find the density of the mystery liquid, you would find the mass of 1 ml of the liquid.

11. Find the density of the mystery liquid, using three different pairs of mass / volume values from Reg's and Bea's data. Do all your answers agree? Explain.

12. Is the relationship between volume and mass a proportional relationship? Explain.

The Mystery Grows

Gabe's data:

volume	mass
10 ml	66 g
20 ml	82 g
40 ml	114 g
60 ml	146 g

13. a. On the same axes, plot Gabe's data.
    b. In Gabe's table, if you double the volume, does the mass double? Check this in two cases.

14. If you add 20 ml, how much mass are you adding? Is this consistent with what you learned from Reg's and Bea's data?

15. a. *According to Gabe's graph*, what is the mass of 0 ml of the liquid? Does this make sense?
    b. What might be the real meaning of the y-intercept on Gabe's graph? Did Gabe make a mistake? Explain.

16. Divide mass by volume for three different pairs of values from Gabe's data. Do all your answers agree? Will that method work to find the density of the liquid?.

17. Are the numbers in Gabe's table examples of a proportional relationship? Explain.

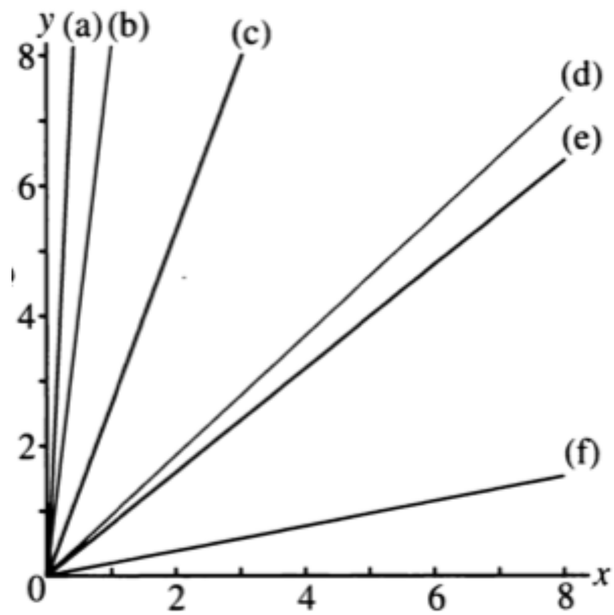
18. Write an equation that expresses mass as a function of volume for Gabe's data.

19. Compare Gabe's graph to Reg's and Bea's. How are they the same, and how are they different?

20. There are number patterns and graph patterns in all the data in this lesson.
    a. What pattern is there in all of Reg's, Bea's, and Gabe's data?
    b. What patterns are only true of Reg's and Bea's data?

Other Substances

21. The graph shows the relationship between mass in grams, and volume in milliliters for some familiar substances. The substances are aluminum, cork, gold, ice, iron and oak. Which substance do you think is represented by each line? Explain.



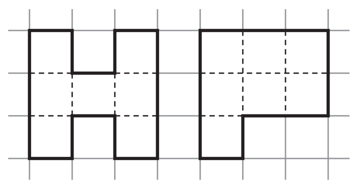
LAB 8.1

Name(s) ______________________

Polyomino Perimeter and Area

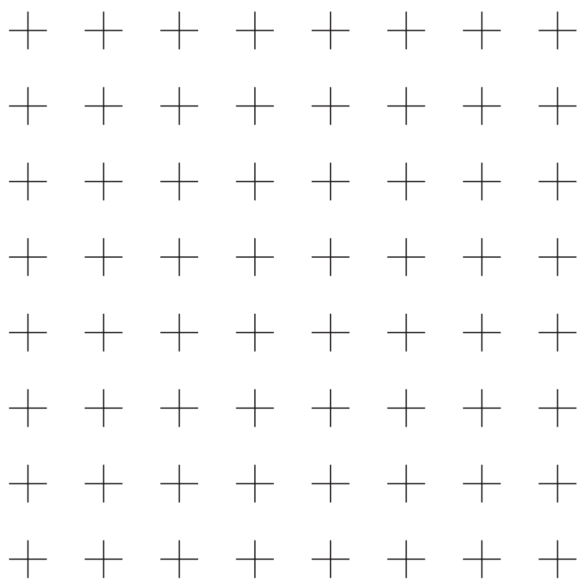
- **Equipment:** 1-Centimeter Grid Paper, interlocking cubes

A polyomino is a graph paper figure whose outline follows the graph paper lines and never crosses itself. We will only consider polyominoes with no holes, such as these:



1. What are the area and perimeter of each polyomino shown above?

2. Find a polyomino with the same area as the ones above, but with a different perimeter. Sketch it in the grid below.



3. Experiment on graph paper or with the help of your interlocking cubes and fill out the following table.

Area	Minimum perimeter	Maximum perimeter
1		
2		
3	8	8
4		
5	10	12
6		
7		
8		
9		
10		
11		
12		
13		

Area	Minimum perimeter	Maximum perimeter
14		
15		
16		
17		
18		
19		
20		
21		
22		
23		
24		
25		
26		

4. Find a formula for the maximum perimeter, P_{max}, for a given area A.

5. Describe a pattern for the minimum perimeter.

6. What would the minimum and maximum perimeters be for the following areas?

a. 49 min. ________ max. ________

b. 45 min. ________ max. ________

c. 50 min. ________ max. ________

d. 56 min. ________ max. ________

Polyomino Perimeter and Area (continued)

Discussion

A. Is it possible to draw a polyomino with an odd perimeter? Explain how to do it, or why it is impossible.

B. While filling out the table, what was your strategy for finding the maximum and minimum perimeters?

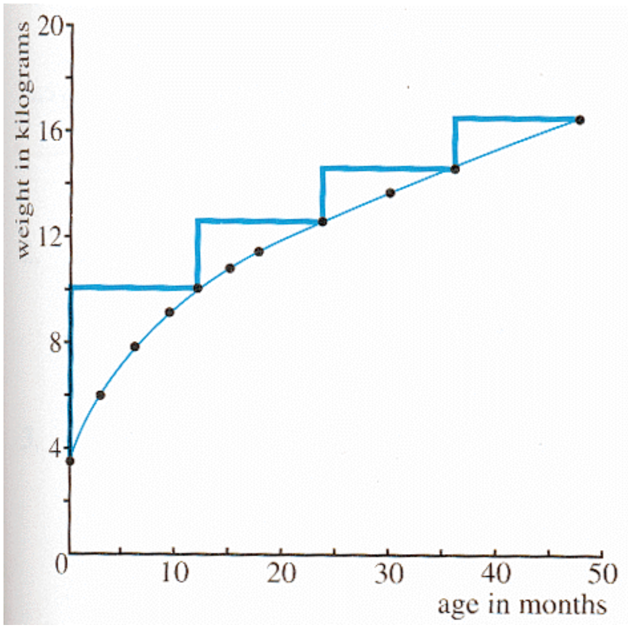
C. Graph minimum and maximum polyomino perimeters with area on the *x*-axis and perimeter on the *y*-axis. What does the graph show about the formula and pattern you found in Problems 4 and 5?

D. Find the polyominoes whose area and perimeter are numerically equal.

E. Explain your strategy for answering the questions in Problem 6.

Weight as a Function of Age

This is a graph of weight as a function of age for Joshua. Four "steps" connect some data points.



Age	Weight (kg)
birth	3.4
3 mos.	5.7
6 mos.	7.6
9 mos.	9.1
12 mos.	10.1
15 mos.	10.8
18 mos.	11.4
2 yrs.	12.6
2.5 yrs.	13.6
3 yrs.	14.6
4 yrs.	16.5

1. Use the data table to find the weight (in kilograms), and the width (in months) of each step. Explain the meaning of these numbers in terms of the *yearly* change in Joshua's weight.

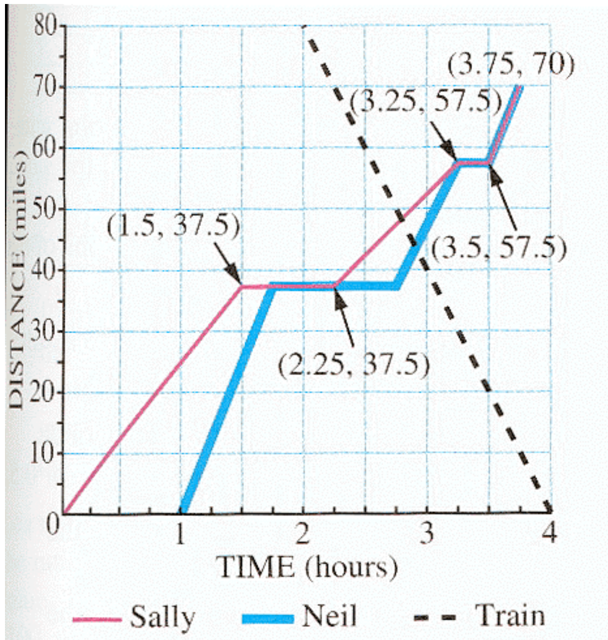
2. Find the average *monthly* weight gain between ages:
   a. two and two and a half
   b. two and a half and three
   c. two and three

3. Joshua's weight grew at a fairly constant monthly rate between ages one and four. Explain how this can be seen:
   a. on the graph.
   b. numerically.

4. However it grew much slower between ages one and four than during his first year. Explain how this can be seen:
   a. on the graph.
   b. numerically.

The Bicycle Trip

Sally is riding her bike on a trip with her bicycle club. She left the staging area in Chapley at 10 am, and took a break at a rest area located about half way to the final destination of Berkhill, 70 miles away. Neil is driving the sweep vehicle, a van with food, water, first aid, and a bicycle rack. The distance-time graph below shows their progress. There are train tracks along the road. The progress of a train is also shown on the graph.



1. Compare Sally and Neil's progress. Who left first? Where did they stop? What happened at the end? What was the total distance covered?

2. Including the origin, the coordinates of six points on Sally's graph are given. Describe her ride between consecutive points.
   a. At what time did each leg of her trip start and end? How far did she ride each time? How long did it take? How long were her breaks?
   b. How fast was she going during each leg of the trip?

3. a. If you were to guess about which part of the trip was downhill or uphill, what would you guess? Why?
   b. How else might one account for the different speeds?

4. How fast did Neil drive in each leg of his trip?

5. Describe the train's progress. Which way was it going? Where and when did it pass Sally and Neil?

6. Where were Sally, Neil, and the train at 12:30pm?

7. At what time were Sally, Neil, and the train 20 miles from the staging area?

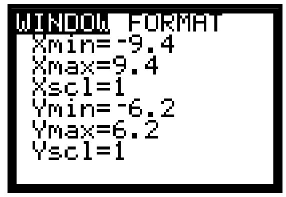
8. The equation of the train's motion is $D = 160 - 40t$.
   a. Choose three points on the train's graph, and check that their coordinates satisfy the equation.
   b. Do any points in Sally's and Neil's graphs satisfy the train's equation? Which ones?

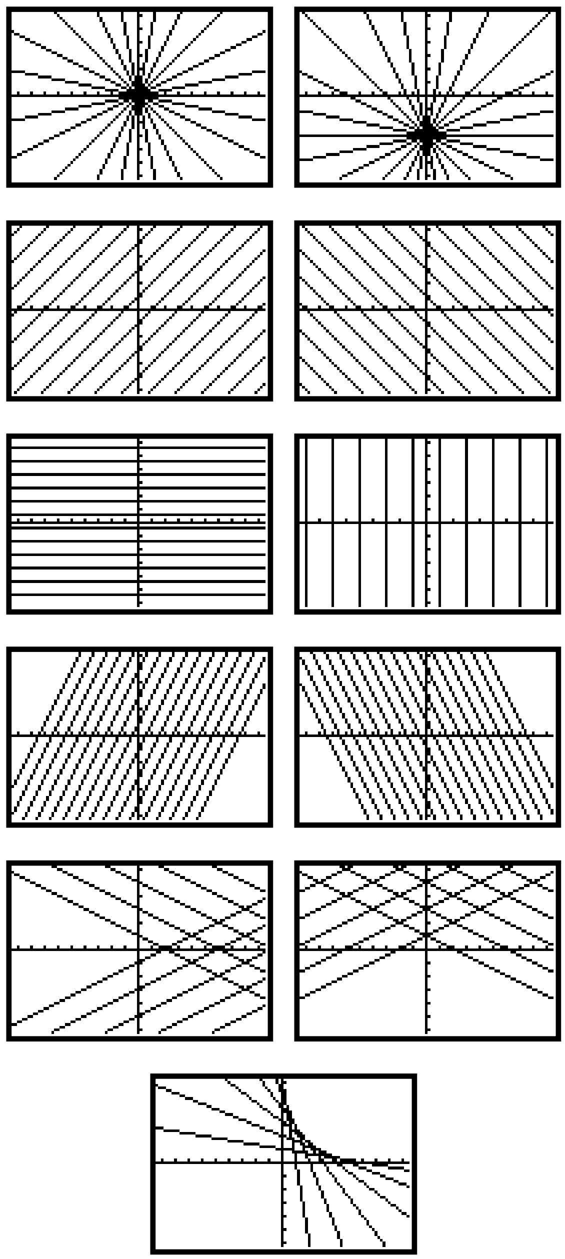
9. **Summary.**
   a. In a distance-time graph like the one above, what does it mean if two points are on the same horizontal line? on the same vertical line?
   b. As you follow someone's progress from left to right on the graph, what is the meaning of a part of a graph that goes up? down? What is the meaning of a horizontal segment? Why is a vertical one impossible?
   c. What is the significance of a point that belongs to the motion graphs of two different people?

10. **Report.** Tell the story of the bicycle trip, using the information you gathered from the graph. You can make guesses about the trip as long as they do not conflict with the given information. Include a graph for Irva, another member of the bicycle club, knowing that she too left at 10 am, and stopped at the rest area.

Make These Designs, TI-83/84

Henri Picciotto





Doing Dishes

Abe needs money, so he is planning to tell his parents he will do the dishes every day, if they would agree to pay him.

He is thinking of asking for $1 the first day, $1.10 on the second day, $1.20 on the third day, and so on, adding ten cents a day.

One Week

1. If his parents agree, how much money will he have made by the end of one week?

Hint: make a table like this one to keep track of your calculations, and then add the numbers:

Day #	Dollars
1	1.00
2	1.10
3	1.20
4	…

His older sister Lara suggests another idea: ask for one cent on the first day, two cents on the second day, four cents on the third day, and so on, doubling the amount each day. Abe wonders whether this is a better idea.

2. If his parents agree, how much money will he have made by the end of one week under Lara's plan?

Day #	Cents
1	1
2	2
3	4
4	…

3. Which is the better plan?

One Month

4. How much money would Abe get paid under each plan on Day #10?

Lara tells Abe that he should figure out how much he would get paid on Day #30, at the end of the month, but Abe feels it's too much work to figure it out.

Lara tells him that there is a shortcut to find those numbers!

To find out how much he would get paid under his plan on Day #10, he has to add $0.10 nine times to the original $1.00. So it's

$$1 + 9 \cdot (0.1)$$

Under Lara's plan, he needs to multiply 1 by 2 nine times. So it's:

$$1 \cdot 2^9$$

5. Do the shortcuts agree with your answer to #4?

6. Use the shortcuts to figure out how much he would be paid under each plan. Be careful!
   a. on Day #20
   b. on Day #30

7. Which is the better plan? Explain.

Linear Growth, Exponential Growth

The daily payment in the first plan is an example of *linear growth*.

8. Why is it called "linear"?

9. Write a formula for the payment on Day #n under the first plan.

The daily payment in Lara's plan is an example of *exponential growth*.

10. Why is it called "exponential"?

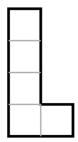
11. Write a formula for the payment on Day #n under Lara's plan.

Perimeter and Area Functions

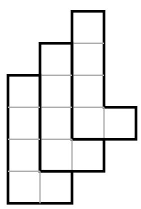
You will need: graph paper.

Pentomino Strips

A pentomino is a shape made of five graph paper squares, connected edge-to-edge. For example, this shape is called the L pentomino:



1. What is the perimeter of the L pentomino? What is its area?



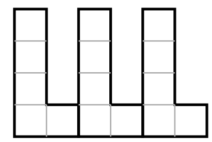
2. Draw a strip of L pentominoes, as shown in the figure above. What are the perimeter and area if you used 3 Ls?

3. Make a table like this, extending it to 7 rows:

Ls	Perimeter	Area
1	…	
2	16	10
3	…	

4. Explain how you would find the perimeter of a 100-L strip without drawing it.

5. How many Ls were used if the perimeter was 92?



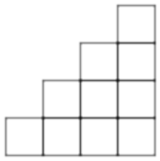
6. Repeat problems 2-5 for an arrangement like the one above.

7. You can use graphs to compare the perimeter patterns for the two pentomino strip arrangements.
   a. Draw a pair of axes. Label the horizontal axis "Number of Ls" and the vertical axis "Perimeter".
   b. Graph all the number pairs from your first table. For example, since the 2-L strip has a perimeter of 16, you would plot the point (2,16).
   c. On the same pair of axes, graph all the number pairs from your second table.
   d. Compare the graphs. How are they the same? How are they different?

8. **Extension:** Repeat the above problems using another pentomino.

Tile Staircases

Imagine you have a supply of square tiles, and you use them to make staircases like this one:



Each row contains one more row than the previous one.

12. a. How many tiles are used in a four-row staircase? That would be the area.
    b. What would be the perimeter of a four-row staircase?

13. Make tables like these for the staircases, extending them to ten rows:

Rows	Perimeter
1	4
2	
3	12

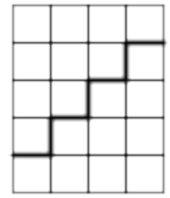
Rows	Area
1	1
2	
3	6

14. Make a graph for each table.

15. **Bonus:** Find a formula for each table

Double-Staircase Rectangles

You can put two copies of a staircase together to make a rectangle, like this:



16. We started with a four-row staircase. What are the length and width of the rectangle we made?

17. What would the length and width be for the rectangle made from two copies of a three-row staircase?

18. Make tables like these, extending them to ten rows:

Rows	Perimeter
1	6
2	
3	14

Rows	Area
1	2
2	
3	12

19. Make a graph for each table.

20. **Bonus:** Find a formula for each table