Topics in finite projective geometry

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This paper is a presentation of several related topics in non-desarguesian projective geometry.

Section (I) will present the definitions, axioms, and notation which will be used in the paper. In particular, Desargues' condition is introduced.

In section (II), collineations (automorphisms of projective planes) and some of their most important properties are introduced. Baer's notion of (p,L)-transitivity is explained, and it is shown to be equivalent to a particular case of Desargues' condition.

In section (III), the projective plane is coordinatized over a ternary field, following M.Hall's method. The algebraic properties of the ternary field are shown to correspond to geometric properties of the plane.

In section (IV), all projective planes are classified in the Lenz-Barlotti types. The classification, in some sense, measures "how desarguesian" the various types of planes are.

In section (V), some finite non-desarguesian planes are constructed, in particular the Hughes planes, which do not have any (p,L)-transitivities.

(I) INTRODUCTION

In this section, we will introduce the basic concepts we will be working with. If the reader is not familiar with them, a good presentation can be found in H pp. 1-8.

Definition : A projective plane is a set of points, of which certain distinguished subsets are called lines, satisfying the following axioms :

PI. Any two distinct points belong to one and only one line.

PII. Any two distinct lines have exactly one point in common.

PIII. There exist four distinct points, no three of which are collinear. (See A p.5, E p.346)

Notation : In this paper, capitals will denote lines. Lower-case letters will denote points. π will denote a projective plane. ab will denote the line through two distinct points a and b. AB will denote the point where two distinct lines A and B meet. For example, we write

(qr)A=x (See fig. 1)

Consider the line of a plane π. Call them "points". Call the points "lines". Say that a "point" belongs to a "line" if and only if the corresponding line contains the corresponding

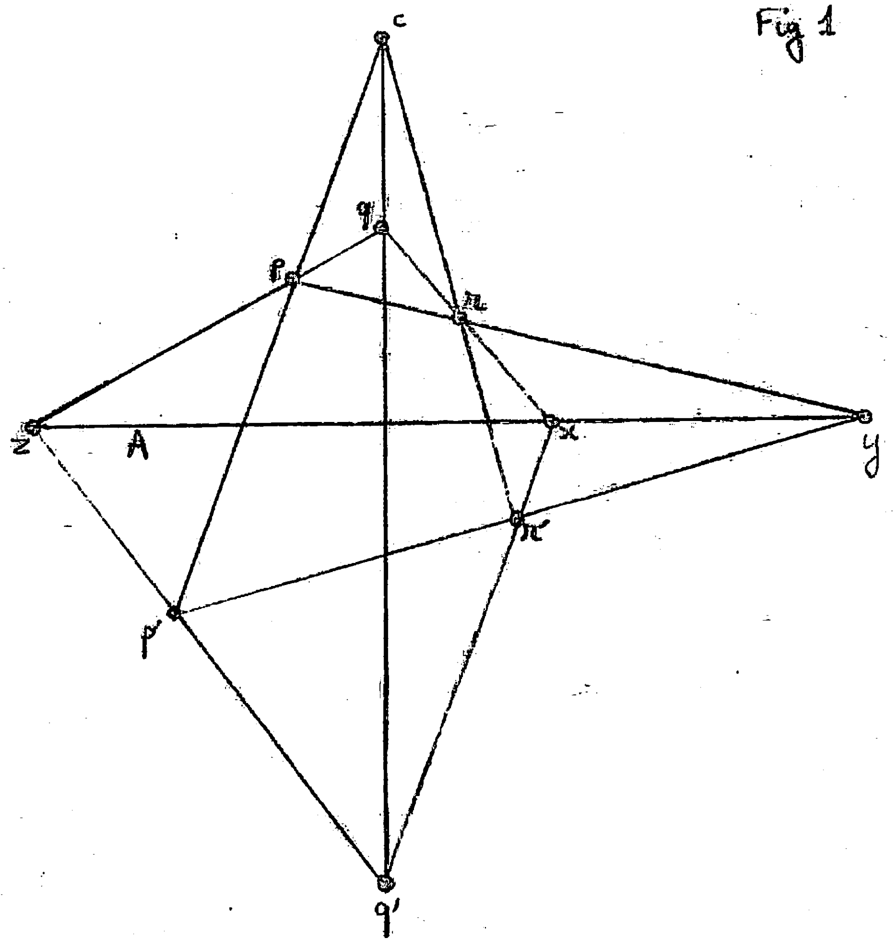


Fig 1

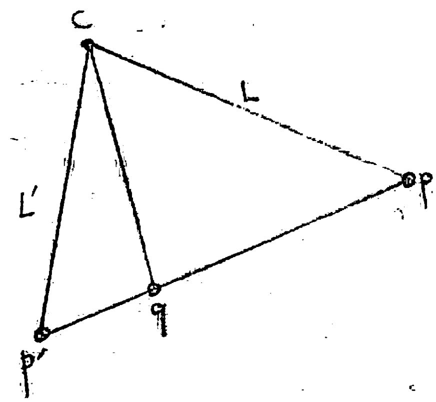


Fig 2

point. The set obtained thus is a projective plane π^* called the dual of π. π and π^* can be shown to be isomorphic. It follows that any true statement about π has a dual statement (obtained by interchanging the words "point" and "line", and interchanging $\in$ and $\ni$) which is also true. This is known as the principle of duality. (See A pp.10-13, E p.347). This principle leads to the notation "pIL" for "p$\in$L", where "I" stands for "incident to". Incidence being a symmetrical relation, this notation is more accurate. However we will use p$\in$L which is more familiar.

Theorem 1 : There are at least three lines through each point. (for proof, see E pp.346-347)

The dual of this theorem is :

Theorem 1* : There are at least three points on each line.

In the following, we will not write down explicitely the duals of theorems. Theorems 1 and 1* will be used very often, without explicit reference to them.

Definition : The points p, q, r, together with the lines pq, qr, and pr, are called a triangle pqr, if they are not collinear. Two triangles pqr and p'q'r' are said to be central if there is a point c (the center) such that $c \in pp'$, $c \in qq'$, and $c \in rr'$. They are said to be axial if there is a line A (the axis) such that $(pq)(p'q') \in A$, $(pr)(p'r') \in A$, and $(qr)(q'r') \in A$. (See fig. 1).

Desargues' condition : Every central couple of triangles is axial.

A plane is called desarguesian if it satisfies Desargues' condition.

Desargues' theorem : A projective plane that can be embedded in a projective 3-space is desarguesian. (For definition of projective 3-space and proof of theorem see H pp.13-16).

If a plane cannot be embedded in a projective 3-space, Desargues' condition does not hold. In this paper we will discuss the geomtric and algebraic conditions that make a plane desarguesian. We will also study some general properties that are charcteristic of non-desarguesian planes.

(II) COLLINEATIONS AND TRANSITIVITY

In this section, we will prove several basic theorems about central collineations, a certain kind of automorphisms of projective planes. We will introduce Baer's notion of (p,L)-transitivity, discovered in 1942, which is extremely important in the theory of projective planes.

Definition : A collineation is an incidence-preserving bijection that maps points of π to points of π, and lines to lines.

Let Δ be the set of all collineations of π. It constitutes a group (Operation : composition of maps. Identity element : the identity map, noted i). Collineations, except i, will be denoted by small greek letters. A point c such that $\alpha(L)=L$ whenever $c \in L$ is called the center of α. Dually, a line A such that $\alpha(p)=p$ whenever $p \in A$ is called the axis of α. (See D pp.118-119).

Theorem 2 : A collineation $\alpha \in \Delta$ has a center if and only if it has an axis. If $\alpha \neq i$, the center and axis are unique.

Proof : Consider $\alpha \neq i$. Assume α has a center c. We must show : α has an axis. Consider two distinct lines L and L' through c. Take $p \in L$ and $p' \in L'$. If p and p' are fixed by α, then

pp' is an axis. Indeed take any $q \in pp'$. We have :

$$\alpha(q) \in \alpha(pp') = \alpha(p)\alpha(p') = pp'.$$

Also $\alpha(q) \in \alpha(cq) = cq$ (since α has center c). So

$$\alpha(q) = (cq)(pp') = q,$$ and q is fixed. (See fig. 2)

If at least one of p,p' is not fixed by α, (say p) consider

$$(pp')(\alpha(p)\alpha(p')) = f.$$

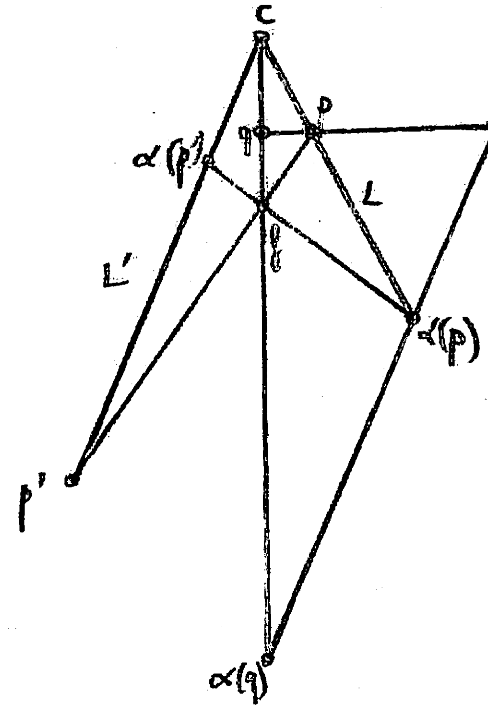
f is fixed, since its image must be on $\alpha(pp')$ which is $\alpha(p)\alpha(p')$ and also on cf. If all $q \in cf$ are fixed, cf is the axis and we're done. Otherwise, consider $q \in cf$ such that $\alpha(q) \neq q$. Consider

$$f' = (pq)(\alpha(p)\alpha(q)).$$

f' is fixed. Reasoning as above, we conclude that ff' is an axis. (See fig. 3).

The proof that if there is an axis, then there is a center, is the dual of the above proof. We still have to show that the center and axis are unique if $\alpha \neq i$. Let A and A' be two distinct axes. Take $p \notin A$, $p \notin A'$, $p \neq c$. Draw a line L through p such that $L \neq cp$ and $L \neq p(AA')$ Then LA and LA' are fixed, so L is an axis and p is fixed. Since p was arbitrary, we must have $\alpha = i$, a contradiction. (See fig. 4). We conclude the axis is unique. Dually, the center is unique. This completes the proof of theorem 2.

We will be almost exclusively interested in central collineations, i.e. the ones that



Fig

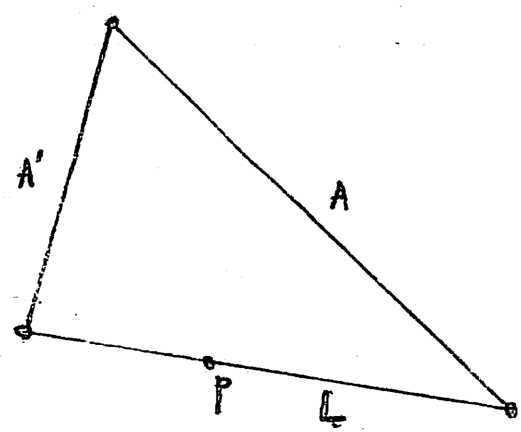


Fig 4

have a center and axis. The following set is a subgroup of Δ :

$$\Delta(c,A)=\{\alpha\in\Delta \mid \alpha \text{ has center } c \text{ and axis } A\}$$

This follows immediately from the uniqueness of the center and axis. Indeed, if α and β $\in\Delta(c,A)$, then clearly $\beta\circ\alpha$ has center c and axis A. (See <u>A</u> p.55).

A remarkable and important fact about central collineations is that they are determined uniquely by their action on any one non-fixed point. More precisely :

<u>Theorem 3</u> : Let $c\in\pi$, $A\in\pi^*$. Let $x,y\in\pi$ be such that $x\neq c$, $y\neq c$, $x\notin A$, $y\notin A$ and $cx=cy$. Then there is at most one $\alpha\in\Delta(c,A)$ such that $\alpha(x)=y$. (See <u>D</u> p.22).

<u>Proof</u> : If there exists such an α, let us show that for a given p, $\alpha(p)$ is uniquely determined. If $p\notin xy$, draw the line px. Let $(px)A=a$. Then

$$\alpha(ax)=\alpha(a)\alpha(x)=ay, \text{ so } \alpha(p)=(ay)(cp).$$

If $p'\in xy$, use a similar construction, with some $p\notin xy$ and $\alpha(p)$ playing the part of x and y respectively. (See fig. 5). This proves theorem 3 and gives us a method to construct $\alpha(p)$ that will be very useful in future proofs.

We are now ready to define (c,A)-transitivity. This concept, due to Baer, is the key to sections

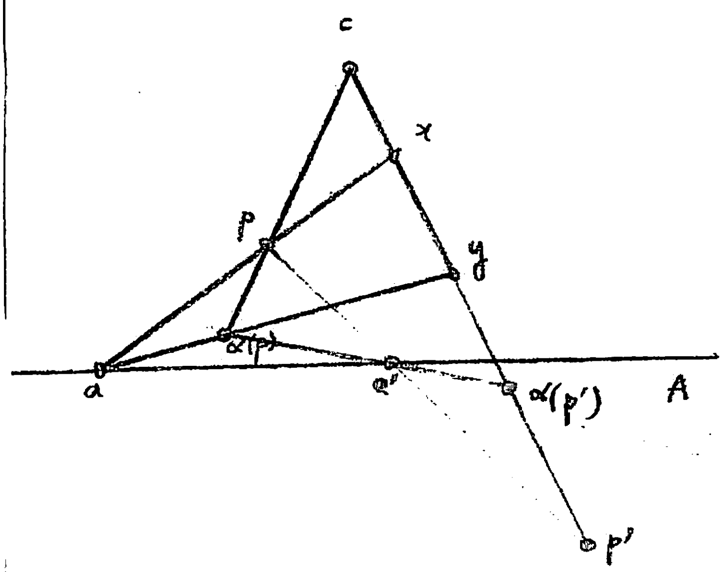


Fig 5

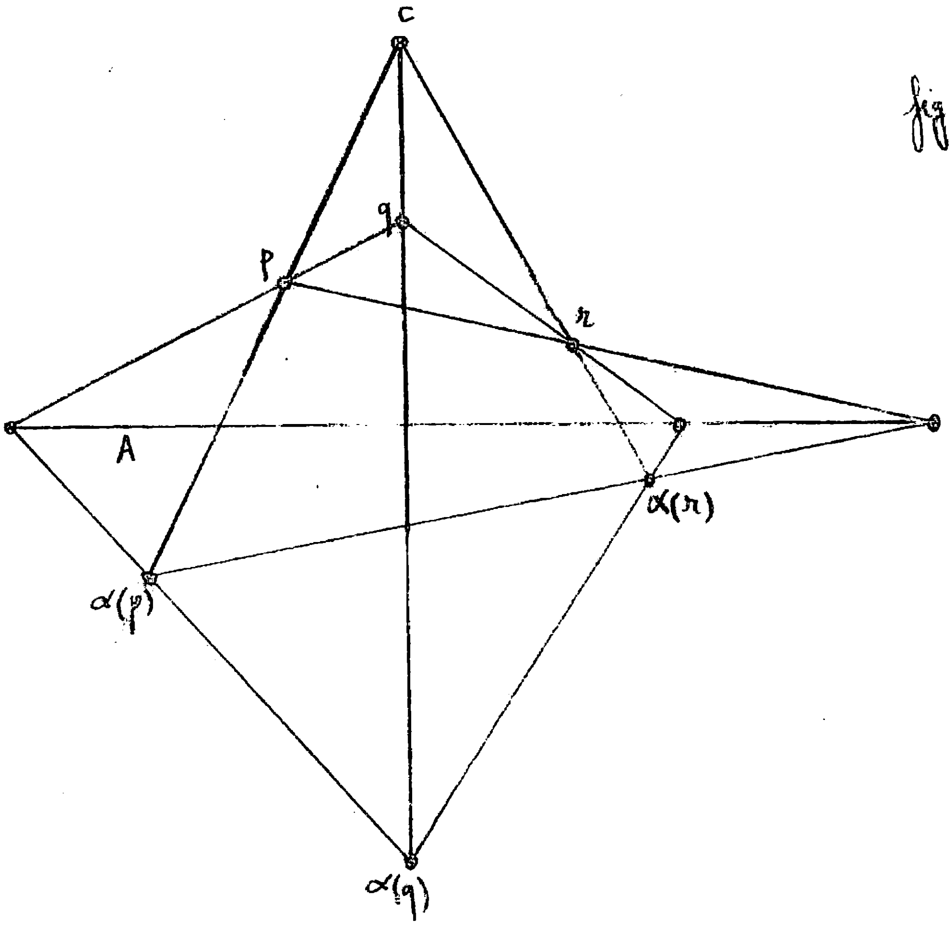


fig 6

(III) and (IV) of this paper.

Definition : π is (c,A)-transitive if Δ contains all possible collineations with center c and axis A, i.e. if the "at most" in the statement of theorem 3 can be replaced by "exactly".

This concept is closely related to special case of Desargues' condition. We say that π is (c,A)-desarguesian if every couple of triangles pqr, p'q'r' with center c and such that

$$(pq)(p'q')\in A \text{ and } (pr)(p'r')\in A$$

is axial. (See fig. 1).

We say that π is (L,A)-transitive if and only if it is (c,A)-transitive for all $c\in L$. Dually, π is (c,p)-transitive if and only if π is (c,A)-transitive for all A such that $p\in A$. (L,A)- and (c,p)-desarguesian are defined in a similar fashion.

Theorem 4 : π is (c,A)-transitive if and only if π is (c,A)-desarguesian.

(The following proof is based on A pp. 59-62).

Lemma 1 : Let $\alpha\in\Delta(c,A)$. If pqr is a triangle such that $c\notin pq$, $c\notin qr$, $c\notin pr$, and $p\notin A$, $q\notin A$, $r\notin A$, then the triangles pqr and $\alpha(p)\alpha(q)\alpha(r)$ have center c and axis A.

Proof of lemma : Since c is the center of α, we have $c\in p\alpha(p)$, $c\in q\alpha(q)$ and $c\in r\alpha(r)$. Since

all points of A are fixed by α, we have ;
$\alpha((pr)A)=\alpha(pr)\alpha(A)=(\alpha(p)\alpha(r))A=(pr)A$, So :
$(pr)(\alpha(p)\alpha(r))\in A$. Similarly, $(pq)(\alpha(q)\alpha(r))\in A$, and $(qr)(\alpha(q)\alpha(r))\in A$. We conclude that the triangles have center c and axis A, and lemma 1 is proved. (See fig. 6).

Proof of theorem : Assume π is (c,A)-transitive. Consider two triangles pqr and p'q'r', with center c, such that $z=(pq)(p'q')\in A$, and $y=(pr)(p'r')\in A$. We must prove that $x=(qr)(q'r')\in A$. (See fig. 1). Consider $\alpha\in\Delta(c,A)$ such that $\alpha(p)=p'$. Then :

$\alpha(q)=\alpha((zp)(cq))=(\alpha(z)\alpha(p))(\alpha(cq))=(zp')(cq)=q'$.
Similarly, $\alpha(r)=r'$. We conclude, by the lemma, that the triangles are axial, and π is (c,A)-desarguesian.

Now assume that π is (c,A)-desarguesian. Consider any x, y, such that $x\neq c$, $y\neq c$, $x\notin A$, $y\notin A$, and $c\in xy$. We must construct $\alpha\in\Delta(c,A)$ such that $\alpha(x)=y$. We can do it as in the proof of theorem 3. But we do not know that the α obtained thus is a collineation. To prove it is, we need the following facts :

a) Well-definition : no matter which $p\notin xy$ is used to define $\alpha(p')$ for $p'\in xy$, we get the same result.

b) Bijection : this is obvious from the construction of α, and (a).

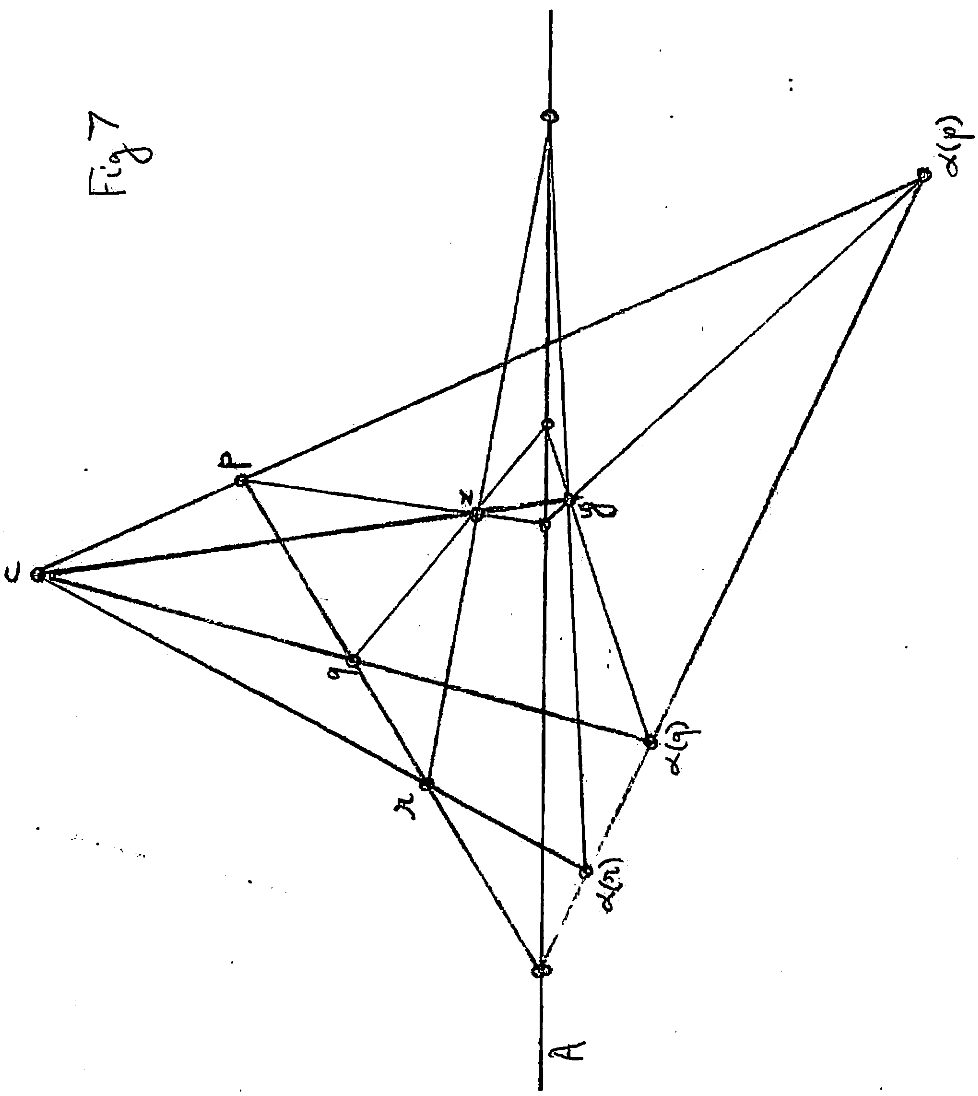
c) α takes collinear points into collinear points.

To prove (a) and (c), we need to know that α takes collinear points of π-xy into collinear points. Consider three collinear points of π-xy, p, q, and r. Construct $\alpha(p)$, $\alpha(q)$, $\alpha(r)$, as in the proof of theorem 3. We must show they are collinear. Consider the triangles pqx and $\alpha(p)\alpha(q)y$. They have center c. By construction, (fig. 7), we have $(qx)(\alpha(q)y) \in A$ and $(px)(\alpha(p)y) \in A$. Since the plane is (c,A)-desarguesian, we conclude that $(pq)(\alpha(p)\alpha(q)) \in A$. Similarly, we can show that $(qr)(\alpha(q)\alpha(r)) \in A$. But pq=qr, so $\alpha(p)\alpha(q)$ and $\alpha(q)\alpha(r)$ meet at $(pq)A=(\alpha(p)\alpha(q))A$. So $\alpha(p)$, $\alpha(q)$ and $\alpha(r)$ are collinear.

Proof of (a) : Consider any p, q in π-xy. Construct $\alpha(p)$ and $\alpha(q)$ as in the proof of theorem 3. Take $p' \in xy$. Draw the lines pp' and qp'. They intersect A respectively at a and b. Define $\alpha(p')=(a\alpha(p))(xy)$ and $\beta(p')=(b\alpha(q))(xy)$. We must show that $\alpha(p')=\beta(p')$. This can be done with no difficulty by using the (c,A)-desarguesian property on the triangles pqp' and $\alpha(p)\alpha(q)\alpha(p')$. (See fig. 8).

Proof of (c) : We have already shown that α takes collinear points of π-xy to collinear points. Clearly $\alpha(xy)=xy$. We are left with the

Fig 7



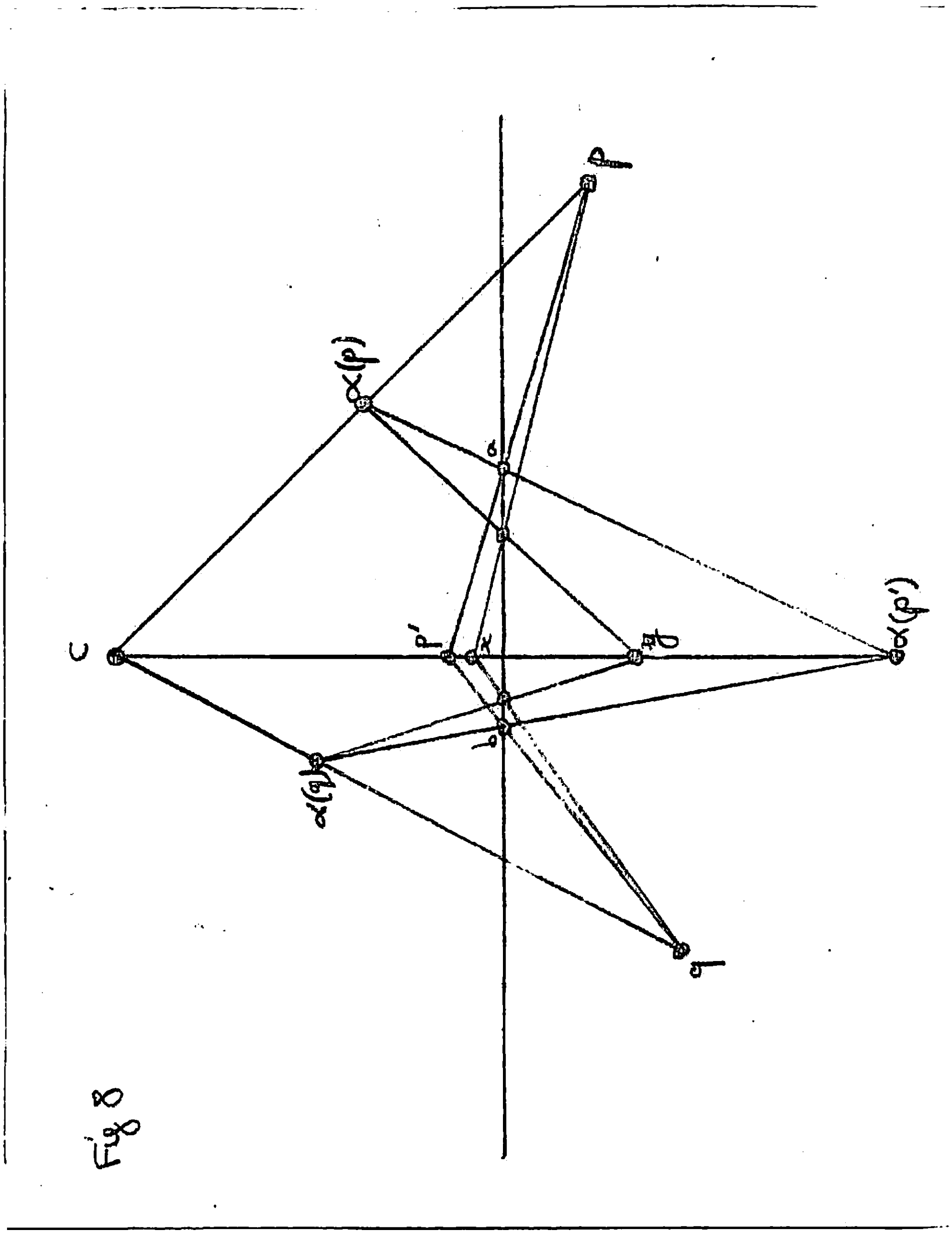


Fig 8

case of three points, one of which is on xy. The result follows immediately from the well-definition of α. Indeed take three collinear points p, q, r, with $p \notin xy$, $r \in xy$. Construct $\alpha(p)$ and $\alpha(q)$. Then construct $\alpha(r)$ using p and then using q. The well definition of α implies $\alpha(p)$, $\alpha(q)$, $\alpha(r)$ are collinear. (See fig. 9). This ends the proof of theorem 4.

This theorem is extremely important, because it tells us what it takes for a plane to be desarguesian. A plane is desarguesian if and only if it has all possible central collineations.

We will conclude this section by proving one more result, that will be useful in sections (III), (IV) and (V).

Theorem 5 : If π is (c,A)-transitive, and $\alpha \in \Delta$, then π is $(\alpha(c),\alpha(A))$-transitive. (See D p. 123).

This result follows readily from the lemma :

Lemma 2 : $\alpha o \Delta(c,A) o \alpha^{-1} = \Delta(\alpha(c),\alpha(A))$.

Proof : Consider $\beta \in \Delta(c,A)$. What are the fixed points of $\alpha o \beta o \alpha^{-1}$?

$\alpha o \beta o \alpha^{-1}(\alpha(a)) = \alpha(\beta(a)) = \alpha(a)$ for all $a \in A$.

So $\alpha(A)$ is the axis. Dually, $\alpha(c)$ is the center, and we conclude that $\alpha o \beta o \alpha^{-1} \in \Delta(\alpha(c),\alpha(A))$.

We can show inclusion in the other direction

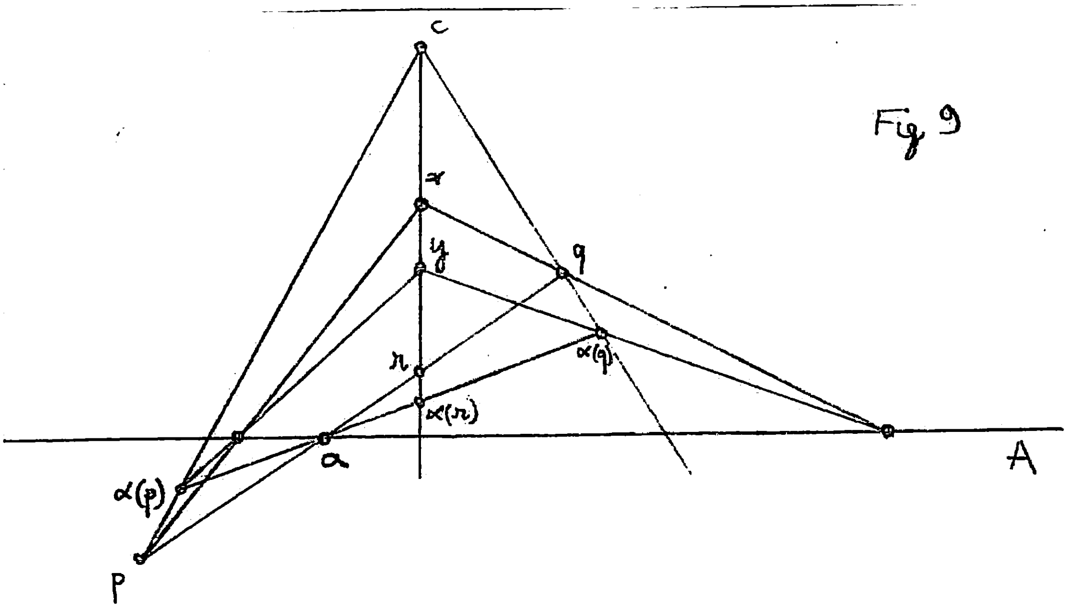


Fig 9

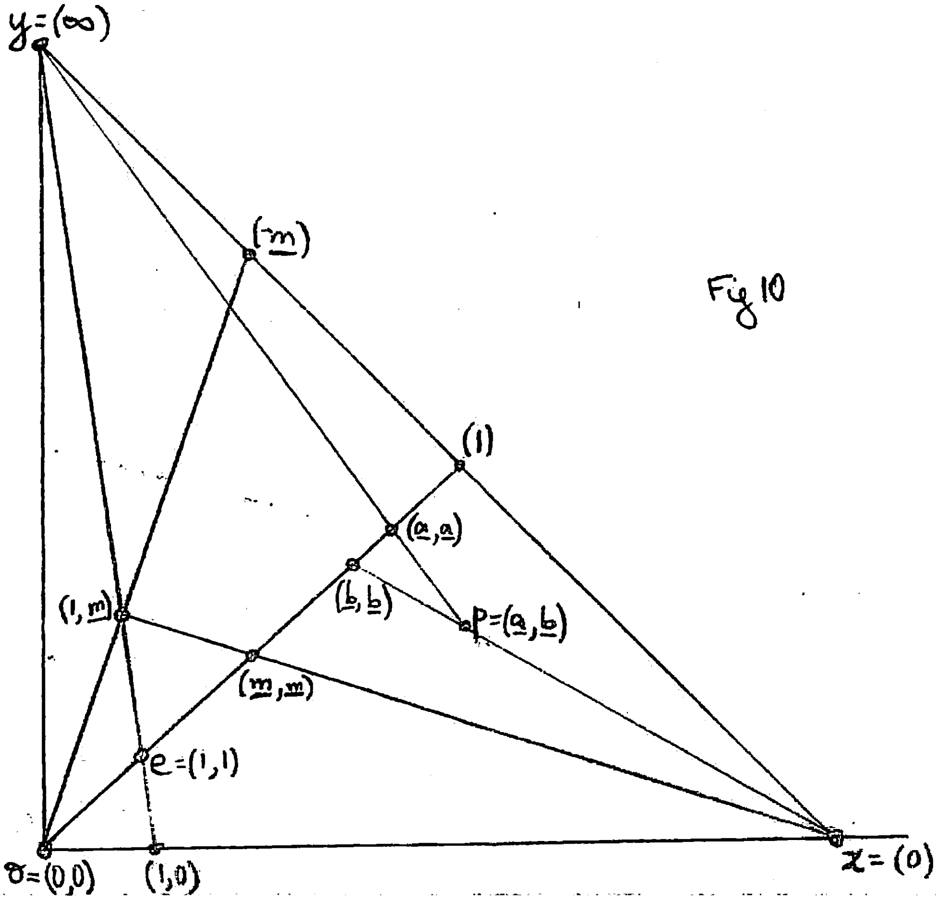


Fig 10

by using the same reasoning. This ends the proof of lemma 2.

(III) COORDINATES AND TERNARY FIELD

In this section, we will bridge the gap between algebra and projective geometry. This is done by introducing coordinates in the projective plane, following M.Hall's approach. (We will follow E pp.353-356 ; see also A pp. 44-51 and D pp. 127-128). Choose four points, no three of which are collinear. Call them o, e, x and y. Assign the coordinates (0,0) to o, (1,1) to e, (1) to (oe)(xy). To other points of oe assign coordinates (b,b), taking different symbols b for different points. For a point p∉xy, let (xp)(oe)=(b,b) and (yp)(oe)=(a,a). Then assign coordinates (a,b) to p. This rule reassigns the same coordinates to points of oe. Assign the coordinate (m) to the point ((0,0)(1,m))(xy). And finally assign the coordinate (∞) to y. (See fig. 10).

Let us define equations for the lines of π, except xy. A line through y other than xy will have the property that all its points (x,y) other than y have the same x-coordinate, say x=c. We take this equality for the equation of the line. Any line not through y will intersect xy at some point (m) and oy at some point (0,b). If q=(x,y) is a point on this line we define a ternary operation on the coordinate system : $\underline{y}=\underline{x}.\underline{m}o\underline{b}$, and take this as the equation of the line. In particular we define :

$\underline{x}+\underline{b}=\underline{x}.\underline{1}o\underline{b}$ and $\underline{x}\underline{m}=\underline{x}.\underline{m}o\underline{0}$. (See fig. 11).

The coordinate system together with the ternary operation is called a ternary field (or planar ternary ring). It will be denoted by ℓ. The ternary operation satisfies the following laws :

a) $\underline{0}.\underline{m}o\underline{c}=\underline{a}.\underline{0}o\underline{c}=\underline{c}$

b) $\underline{1}.\underline{m}o\underline{0}=\underline{m}.\underline{1}o\underline{0}=\underline{m}$

c) Given $\underline{a}$, $\underline{m}$, $\underline{c}$, there exists exactly one $\underline{z}$ such that $\underline{a}.\underline{m}o\underline{z}=\underline{c}$.

d) Given $\underline{m}\neq\underline{m}'$, $\underline{b}$, $\underline{b}'$, there exists exactly one $\underline{x}$ such that $\underline{x}.\underline{m}o\underline{b}=\underline{x}.\underline{m}'o\underline{b}'$.

e) Given $\underline{a}\neq\underline{a}'$, $\underline{c}$, $\underline{c}'$, there exists exactly one pair $\underline{m}$, $\underline{b}$ such that $\underline{a}.\underline{m}o\underline{b}=\underline{c}$ and $\underline{a}'.\underline{m}o\underline{b}=\underline{c}'$.

<u>Proof</u> : (a) and (b) are immediate. (c) says that the line joining ($\underline{m}$) and ($\underline{a},\underline{c}$) intersects oy in a unique point ($\underline{0},\underline{z}$), (Axiom PII). (d) and (e) follow similarly from the axioms.

(a),...,(e) imply that ℓ is a loop with respect to addition. This loop will be denoted by ℓ^+. $\ell-\{\underline{0}\}$ is a loop with respect to multiplication, which will be denoted by $\ell^\bullet$.

The following theorem explains the rela-

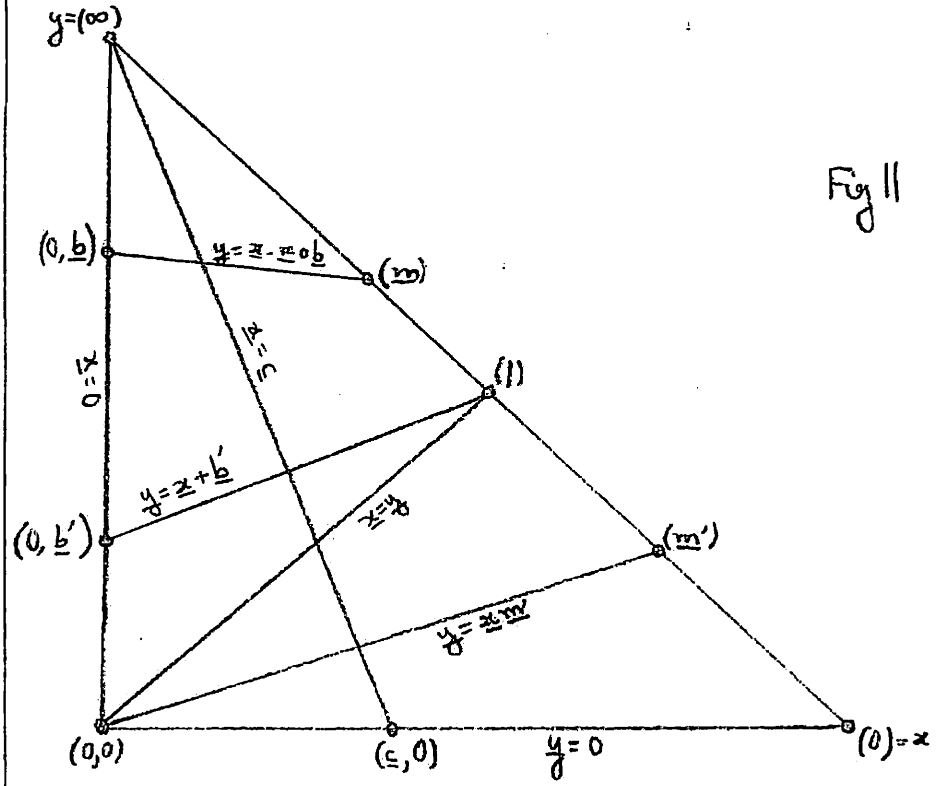

y=(∞)
(0, b)
y = x - x∘b
(m)
x = c
x = 0
(1)
y = x + b′
x = y
(0, b′)
(m′)
y = x m′
(0,0)
(c, 0)
y = 0
(0) = x

Fig 11

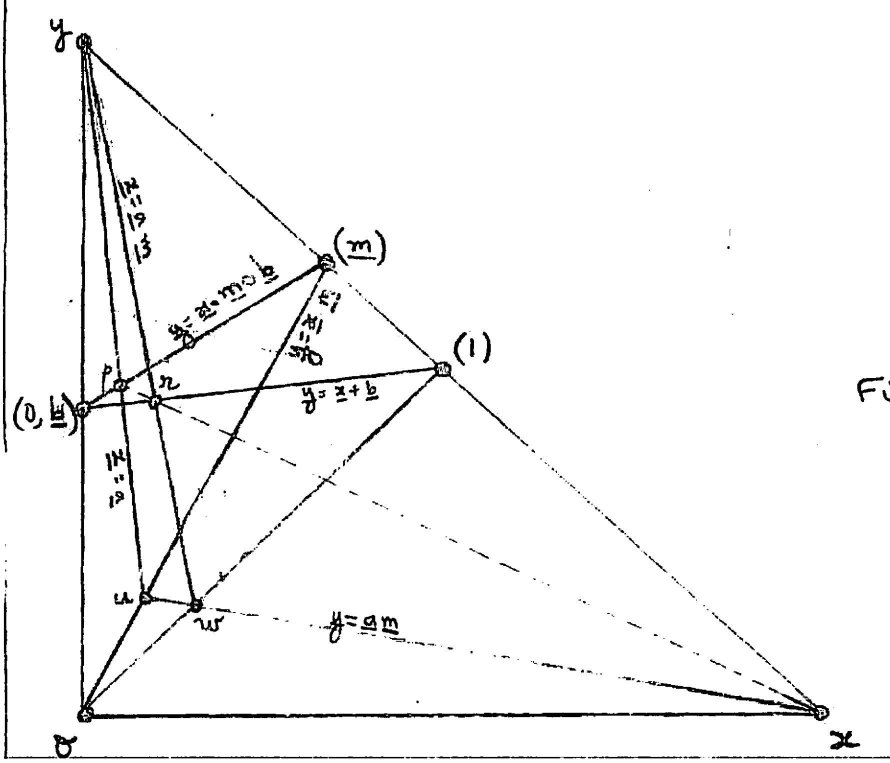

y
(m)
y = x∘m∘b
(1)
(0, b)
y = x + b
u
w
y = a m
o
x

Fig 12

tionship between π and ℓ.

Theorem 6 : Let ℓ be determined by o, e, x, y, and ℓ' be determined by o', e', x' and y'. Then ℓ and ℓ' are isomorphic if and only if there exists $\alpha \in \Delta$ such that $\alpha(o)=o'$, $\alpha(e)=e'$, $\alpha(x)=x'$ and $\alpha(y)=y'$.

Outline of proof : (For full proof, see A p.52). Assume the existence of α. Define a map $\$:\ell \longrightarrow \ell'$ as follows : $(\$(\underline{a}))=\alpha((\underline{a}))$, where $(\underline{a})$ on the right-hand side is a coordinate with respect to ℓ, and the left-hand side is a coordinate with respect to ℓ'. This is possible because α is a bijection between xy and x'y'. Verify \$ is an isomorphism.

Conversely, assume \$ is an isomorphism between ℓ and ℓ'. Define α such that

$$\alpha((\underline{a},\underline{b}))=(\$(\underline{a}),\$(\underline{b})),\ \alpha((\underline{m}))=(\$(\underline{m})),\ \alpha(y)=y'.$$

Verify that α is a collineation. (Coordinates on the left-hand side of the equalities are with respect to ℓ, on the right, to ℓ'.) End of the outline of the proof of theorem 6.

Given a ternary field ℓ, with the properties (a),...,(e) we can construct a projective plane with points $(\underline{a},\underline{b})$, $(\underline{m})$ and (∞), where $\underline{a}$, $\underline{b}$, and $\underline{m}$ range over the elements of ℓ. Indeed let $x=(\underline{0})$ and $y=(\infty)$. The line xy contains y and all the points (m) and no others. For each $c \in \ell$, we have a line x=c consisting of the point y, and the points (c,y) where y ranges over all the elements of ℓ. The point (m) and the points (x,x.mob) are the points of the line y=x.mob, for each pair m, b. The properties (a),...,(e) can be used to prove that the three axioms for projective planes hold. Specifically :

BIII. x, y, (0,0) and (1,1) are such that no three of them are collinear.

For the other two axioms numerous cases must be considered. We will consider a typical one for each axiom.

PI. : (m) and (a,c) belong to the line y=x.moz where z is the unique element of ℓ such that a.moz=c. Moreover, $(m) \notin [x=b]$ for all b, and $(a,c) \notin xy$. We conclude the two points (m) and (a,c) belong to a unique line. The other cases are : (m),(n) ; (a,c), (b,d) ; (∞), (m) ; (∞), (a,c). They do not present any difficulty.

PII. Consider two distinct lines y=x.mob and y=x.mob'. They have the point (m) in common, but no other point. Indeed, (m) is the only point with a single coordinate on either of these lines. Moreover, if they both contained (a,c), we would have a.mob=c=a.mob', contradicting property (c). The other cases are : x=c, x=c' ; y=x.mob,

$y=x.m'ob'$; $y=x.mob$, xy ; $x=c$, xy ; $y=x.mob$, $x=c$. They do not present any difficulty.

This ends the proof that the above construction yields a projective plane. We will refer to it in section (V), where it will be used to construct some finite non-desarguesian planes.

It follows from the above that there is a canonical correspondence between projective planes and ternary fields. We will now show how further algebraic properties of the ternary field correspond to various transitivities in the plane. Here are some such properties (see D pp. 129-130) :

I : $a.mob=am+b$ (ℓ is said to be linear).

II : addition is associative (ℓ^+ is a group).

III : multiplication is associative ($\ell^\bullet$ is a group).

IV : $(x+y)z=xz+yz$ (right distributivity).

V : $x(y+z)=xy+xz$ (left distributivity).

VI : $x^2y=x(xy)$ and $xy^2=(xy)y$.

VII : $xx'=1 \implies x(x'y)=y$ (left inversive property).

VIII : $yy'=1 \implies (xy)y'=x$ [illegible]
(right inversive property).

IX : multiplication is commutative.

Note that I, II, and IV (or I, II and V) imply that addition is commutative. (For proof, see A p.69)

Definition : A right (resp. left) quasifield (or Veblen-Wedderburn system) is a ternary field satisfying I, II, and IV (resp. V). A semifield is a ternary field satisfying I, II, IV and V. A right (resp. left) planar nearfield is a ternary field satisfying I, II, III, and IV (resp. V). An alternative field is a ternary field satisfying I, II, IV, V, and VI.

Note that a ternary field satisfying I,..., V is a (not necessarily commutative) field.

We now have all the terminology we need to state the major theorem of this section :

Theorem 7 : (a) π is (y,xy)-transitive if and only if ℓ satisfies I and II. In this case, Δ(y,xy) is isomorphic to ℓ^+.

b) π is (x,oy)-transitive if and only if ℓ satisfies I and III. In this case, Δ(x,oy) is isomorphic to $\ell^{\bullet}$.

c) π is (xy,xy)-transitive if and only if ℓ is a right quasifield.

d) π is (y,y)-transitive if and only if ℓ is a left quasifield.

e) π is (x,y)-transitive if and only if ℓ is a right planar nearfield.

f) π is (xy, xy)-transitive and (oy,oy)-transitive if and only if ℓ satisfies $\underline{I}$, $\underline{II}$, $\underline{IV}$, $\underline{V}$, and $\underline{VII}$.

g) π is (x,oy)- and (y,oe)-transitive if and only if ℓ is a (not necessarily commutative) field. (See $\underline{D}$ p.130).

Proof : Gingerich seems to have discovered this theorem in 1945. See $\underline{G}$ pp. 31-58 for a complete proof. Here, we will prove (a), following $\underline{E}$ pp. 360-362 (See also $\underline{A}$ pp. 65-66). The other results can be proved in a similar fashion. However, (e) follows easily from (b) and (c), and (g) follows easily from (d) and (e).

Suppose that π is (y,xy)-transitive. We must show ℓ satisfies $\underline{I}$ and $\underline{II}$. Take points $(\underline{m})$, $(\underline{O},\underline{b})$. Take $p\in[\underline{y}=\underline{x}\cdot\underline{m}o\underline{b}]$. Say $p=(\underline{a},\underline{a}\cdot\underline{m}o\underline{b})$. We have $(yp)((\underline{m})o)=u=(\underline{a},\underline{am})$. Also : $(ux)((\underline{1})o)=w=(\underline{am},\underline{am})$, and $(yw)((\underline{m})(\underline{O},\underline{b}))=r=(\underline{am},\underline{am}+\underline{b})$. (See fig. 12).

Now rx has equation $\underline{y}=\underline{am}+\underline{b}$, so if we can show $p\in rx$ we will have shown that $\underline{am}+\underline{b}=\underline{a}\cdot\underline{m}o\underline{b}$. Take $\alpha\in\Delta(y,xy)$ such that $\alpha(o)=(\underline{O},\underline{b})$. Then the image of $u=(yu)((\underline{m})o)$ is :

$$\alpha(u)=\alpha((\underline{m})o)\alpha(yu)=((\underline{O},\underline{b})(\underline{m}))(yu)=p.$$

Similarly, $\alpha(w)=r$. And we know that $\alpha(x)=x$. Since $u\in wx$, it follows that $p\in rx$. This completes the proof that ℓ satisfies $\underline{I}$.

What is the image of a point $(\underline{a},\underline{c})$? Well : $(\underline{a},\underline{c})=[\underline{x}=\underline{a}][\underline{y}=\underline{c}]$, and $\alpha([\underline{x}=\underline{a}])=[\underline{x}=\underline{a}]$. We must determine $\alpha([y=c])$. (See fig 13). We have $\alpha((\underline{c},\underline{c}))=(\underline{c},\underline{c}+\underline{b})$ since $\alpha((\underline{1})o)=(\underline{0},\underline{b})(\underline{1})$. It follows that $\alpha([y=c])=[y=c+b]$. So :

$$\alpha((\underline{a},\underline{c}))=(\underline{a},\underline{c}+\underline{b}).$$

Now if $\beta\in\Delta(y,xy)$ is such that $\beta(o)=(0,\underline{d})$, we have $\beta((\underline{u},\underline{v}))=(\underline{u},\underline{v}+\underline{d})$. And for $\beta o\alpha$ we find :

$$(\beta o\alpha)(o)=\beta((\underline{0},\underline{b}))=(\underline{0},\underline{b}+\underline{d}). \text{ So :}$$

$$(\beta o\alpha)((\underline{a},\underline{c}))=(\underline{a},\underline{c}+(\underline{b}+\underline{d})). \text{ But :}$$

$\beta(\alpha((\underline{a},\underline{c})))=\beta((\underline{a},\underline{c}+\underline{b}))=(\underline{a},(\underline{c}+\underline{b})+\underline{d})$. Hence :

$\underline{c}+(\underline{b}+\underline{d})=(\underline{c}+\underline{b})+\underline{d}$, and since $\underline{b}$, $\underline{c}$, $\underline{d}$, were arbitrary, addition is associative, and we have proved that ℓ satisfies II.

Conversely, suppose ℓ satisfies I and II. For any $\underline{b}\in\ell$, define a map α such that $\alpha(y)=y$, $\alpha((\underline{m}))=(\underline{m})$, and $\alpha((\underline{a},\underline{c}))=(\underline{a},\underline{c}+\underline{b})$. Let us show α is a collineation. Clearly, $\alpha(xy)=xy$, and $\alpha([\underline{x}=\underline{a}])=[\underline{x}=\underline{a}]$. If $(\underline{a},\underline{c})\in[\underline{y}=\underline{xm}+\underline{t}]$, then $\underline{c}=\underline{am}+\underline{t}$, so $\underline{c}+\underline{b}=(\underline{am}+\underline{t})+\underline{b}=\underline{am}+(\underline{t}+\underline{b})$, which implies that $(\underline{a},\underline{c}+\underline{b})\in[\underline{y}=\underline{xm}+(\underline{t}+\underline{b})]$. This verifies that the image of any point of $\underline{y}=\underline{xm}+\underline{t}$ is on $\underline{y}=\underline{xm}+(\underline{t}+\underline{b})$. So α takes lines to lines in all cases, and therefore is a collineation. But $\alpha\in\Delta(y,xy)$ and $\alpha(o)=(\underline{0},\underline{b})$. Since $\underline{b}$ was arbitrary, we conclude that π is (y,xy)-transitive.

It remains to show that $\Delta(y,xy)$ is isomorphic to ℓ^{+}. If we associate to each α the number $\underline{b}$ such that $\alpha(o)=(\underline{0},\underline{b})$, it is easy to

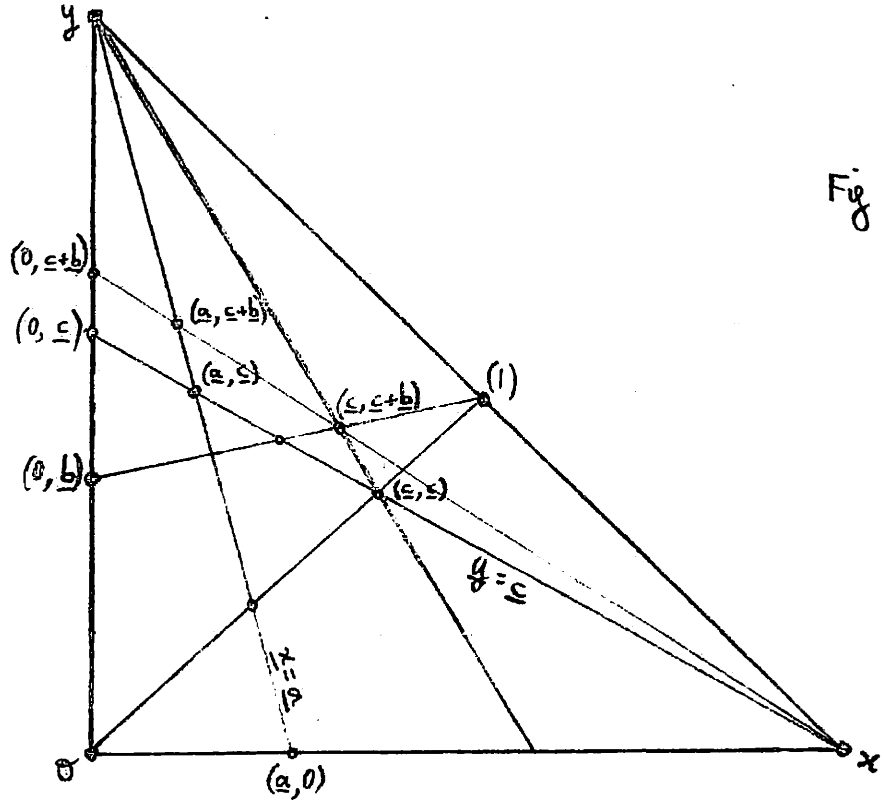


Fig 13

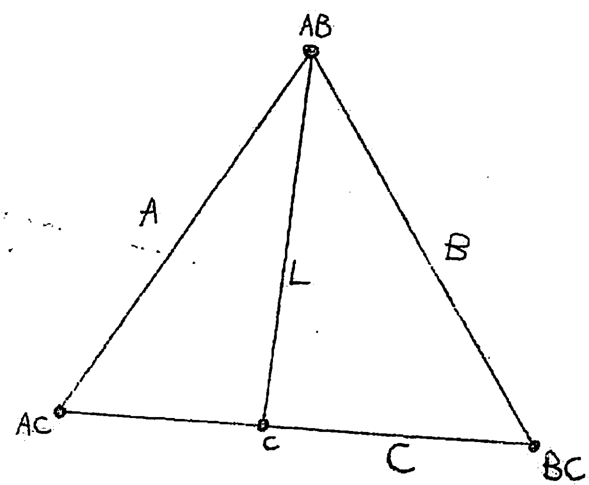


Fig 14

show we have an isomorphism. This ends the proof of theorem 7(a).

Theorem 7 is rather important because it shows the intimate connection between algebra and projective geometry. Not only are geometric facts proved using purely algebraic tools, but also algebraic properties are established through geometric arguments. Notice that as ℓ gets more and more of the properties of a field, π gets closer and closer to a desarguesian plane. This process culminates in theorem 8 below.

Definition : If a plane is (L,L)-transitive for all L, it is called a Moufang plane (after Ruth Moufang who first studied these in 1933).

Theorem 8 : (a) π is a Moufang plane if and only if ℓ is an alternative field.

b) π is desarguesian if and only if ℓ is a (not necessarily commutative) field. (See D p.130).

Proof : (a) The proof of this result requires an algebraic arsenal that we will not even attempt to muster (See E p. 372). However, we will prove the weaker result that a Moufang plane corresponds to an alternative field that satisfies VII and VIII. (We will follow E pp. 366-371).

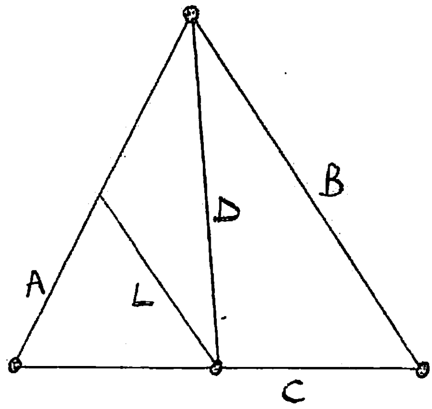
Lemma 3 : If π is (A,A)- and (B,B)-transitive, then it is (L,L)-transitive for all L such that $AB \in L$.

<u>Proof of lemma</u> : Consider $c \in L$, $c \neq AB$. Draw a line C through c, $C \neq L$. Then there exists $\alpha \in \Delta(AC,A)$ such that $\alpha(BC)=c$. It follows that $\alpha(B)=L$. Since π is (BC,B)-transitive, by theorem 5 it is (c,L)-transitive. This is true for any L such that $AB \in L$ and any $c \in L$, so the lemma is proved. (See fig. 14). It is the use of this lemma in the proof of theorem 8(a) that accounts for the fact that it is much more powerful than any one of the results in theorem 7.

<u>Corollary</u> : If π is (A,A)-, (B,B)- and (C,C)-transitive, where A, B, C are not concurrent, then π is a Moufang plane.

<u>Proof of corollary</u> : Consider any line L. We must show (L,L)-transitivity. Draw (AB)(LC)=D. By the lemma, π is (D,D)-transitive. By the lemma π is (L,L)-transitive. (See fig. 15). End of proof of corollary. Now let us go back to the proof of theorem 8 (a).

Assume ℓ satisfies <u>I</u>, <u>II</u>, <u>IV</u>, <u>V</u>, <u>VI</u>, <u>VII</u>, and <u>VIII</u>. Then by theorem 7 (f) and lemma 3, π is ([$\underline{x}=\underline{c}$],[$\underline{x}=\underline{c}$])-transitive for all $\underline{c} \in \ell$, as well as (xy,xy)-transitive. Let us find a collineation that moves y. α defined as follows will do the job.: $\alpha((\underline{a},\underline{b}))=(\underline{b},\underline{a})$, $\alpha((\underline{m}))=(\underline{m}^{-1})$ for $\underline{m} \neq \underline{0}$, and $\alpha(x)=y$, $\alpha(y)=x$. α is a collineation since $\alpha([\underline{x}=\underline{c}])=[\underline{y}=\underline{c}]$, $\alpha([\underline{y}=\underline{xm}+\underline{b}])=[\underline{y}=\underline{xm}^{-1}-\underline{bm}^{-1}]$



Fig

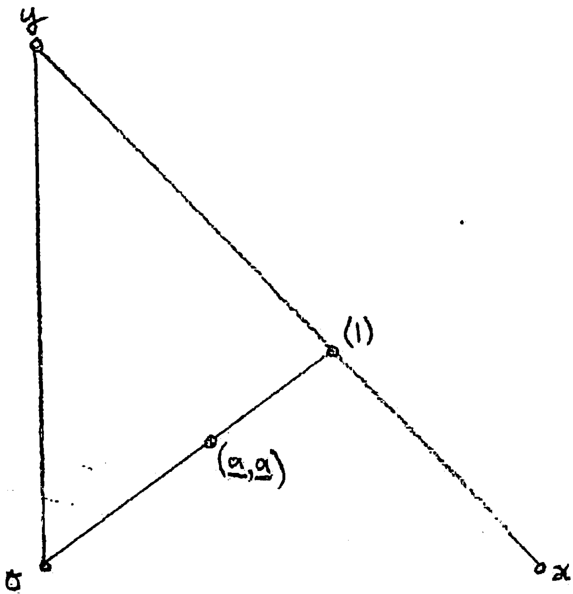


Fig 16

for $\underline{m}\neq 0$, and $\alpha(xy)=xy$. We have $\alpha\in\Delta((\underline{1}),oe)$.
Let us show π is (oe, oe)-transitive. Pick any point $(\underline{a},\underline{a})$. There exists $\beta\in\Delta(o,oy)$ such that $\beta((\underline{1}))=(\underline{a},\underline{a})$. By theorem 5, π is $((\underline{a},\underline{a}),oe)$-transitive. (See fig. 16). But $(\underline{a},\underline{a})$ was arbitrary, so π is (oe, oe)-transitive, and by the preceding corollary π is a Moufang plane.

Now assume that π is a Moufang plane. By theorem 7 (f), ℓ must satisfy I, II, IV, V and VII. Let us show it satisfies VIII. Consider $\alpha\in\Delta(o,ox)$ such that $\alpha(y)=(\underline{0},-\underline{1})$. Then it is easy to show that $\alpha([\underline{x}=\underline{a}])=[\underline{y}=\underline{x}\underline{a}^{-1}-\underline{1}]$. It follows that $\alpha((\underline{1},\underline{1}-\underline{ab}))=((\underline{ab})^{-1},(\underline{ab})^{-1}-\underline{1})$, and therefore $\alpha([\underline{y}=\underline{1}-\underline{ab}])=[\underline{y}=(\underline{ab})^{-1}-\underline{1}]$. It follows also that $\alpha((\underline{a},\underline{1}-\underline{ab}))=(\underline{b}^{-1},\underline{b}^{-1}\underline{a}^{-1}-\underline{1})$, and therefore $\alpha([\underline{y}=\underline{1}-\underline{ab}])=[\underline{y}=\underline{b}^{-1}\underline{a}^{-1}-\underline{1}]$. Comparing the images of $\underline{y}=\underline{1}-\underline{ab}$, we conclude :

$(\underline{ab})^{-1}=\underline{b}^{-1}\underline{a}^{-1}$. And since by VII, $\underline{b}^{-1}=\underline{a}(\underline{a}^{-1}\underline{b}^{-1})$, we find : $\underline{b}=(\underline{b}^{-1})^{-1}=(\underline{a}(\underline{a}^{-1}\underline{b}^{-1}))^{-1}=(\underline{a}^{-1}\underline{b}^{-1})^{-1}\underline{a}^{-1}$
$=(\underline{ba})\underline{a}^{-1}$.

So ℓ satisfies VIII. Since VI is an algebraic consequence of the other laws that ℓ satisfies, (see E pp. 369-370), this ends the proof of the weak version of theorem 8 (a).

Proof of (b) : (We will follow A, p.77). Assume first that π is desarguesian. By theorem 7 (d), (e), we conclude that ℓ satisfies I,...,V.

Now assume that one of π's coordinate systems satisfies I,...,V. Then by a well-known result about planes that can be coordinatized by a field, there exists a collineation taking o, e, x, y to any other four points (no three of which are collinear). (See A p. 42, H p.93). By theorem 6, this implies that every coordinate system ℓ for π satisfies I,...,V. We want to show that π is desarguesian, i.e. (c,A)-desarguesian for any c and any A. Take an arbitrary point-line pair (c,A). If c∈A, choose o,e,x,y such that c=y and A=xy. If c∉A, choose o, e, x, y such that c=x and A=oy. Then by theorem 7 (a) and (b) we conclude that π is (c,A)-desarguesian. End of proof of theorem 8.

If the reader's esthetic sense is bothered by the "not necessarily commutative" in the statement of theorem 8 (b), he should not despair. An algebraic theorem of Wedderburn's (see A pp. 76-77) shows that in the finite case, IX is a consequence of II,...,V. It follows that :

Theorem 8 (b') : Assume π is finite. Then π is desarguesian if and only if ℓ is a field.

Moreover, in the general case, ℓ being a field corresponds to a geometric condition that is stronger than Desargues' condition, but no less pleasant : Pappus' condition.

<u>Pappus' condition</u> : Consider two lines L and L' and six points a, b, c, a', b', c', distinct from LL' and from each other, such that a, b, c ∈L and a', b', c' ∈L'. Then (ab')(a'b), (ac')(a'c) and (bc')(b'c) are collinear. (See fig. 17).

<u>Theorem 8 (c)</u> : π satisfies Pappus' condition if and only if ℓ is a field. (For proof see <u>A</u> pp. 78-82. For more about Pappus' condition, see <u>D</u> pp. 157-161).

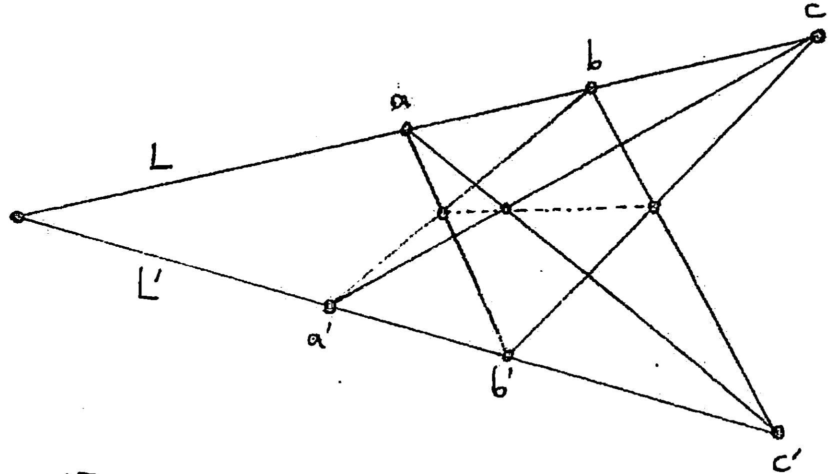


fig 17

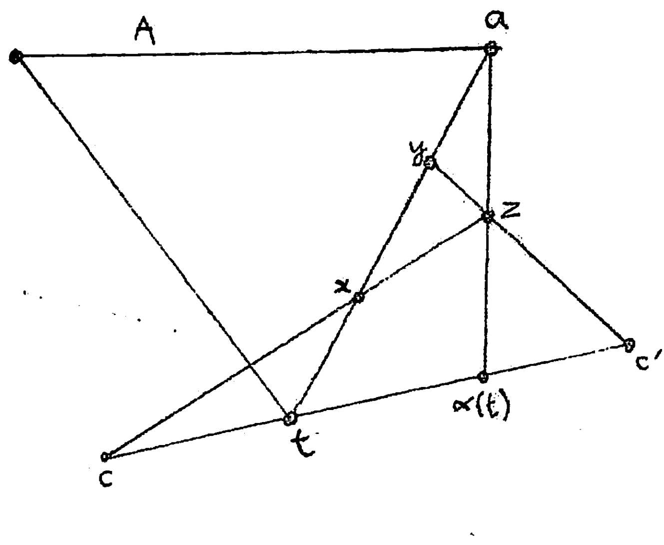


Fig 18

(IV) THE LENZ-BARLOTTI CLASSIFICATION

The classification, discovered by Lenz and refined by Barlotti in 1957, distinguishes the possible types of projective planes by describing their transitivities. We will prove some preliminary results before getting to the classification itself. The proof of the classification requires showing that certain transitivities imply certain other ones. For example theorem 5 and lemma 3 give us ways to do this. Theorems 9 through 12 below will have the same purpose. (Throughout this section we will mainly follow B).

Theorem 9 : If π is (c,A)-transitive and (c',A)-transitive, then it is (cc',A)-transitive. (See D p. 123).

Proof : Case I : Assume c or $c' \notin A$. Consider $t \in cc'$. We must show that π is (t,A)-transitive. Take x, y such that $t \in xy$, $x \neq t$, $y \neq t$, $x \notin A$, $y \notin A$. We must show : there exists $\theta \in \Delta(t,A)$ such that $\theta(x)=y$. Define $z=(cx)(c'y)$. There exists $\alpha \in \Delta(c,A)$ and $\alpha' \in \Delta(c',A)$ such that $\alpha(x)=z$ and $\alpha'(z)=y$. (See fig. 18). Define $\theta=\alpha' \circ \alpha$. θ is a collineation. Moreover it fixes all the points of A. So it has an axis and therefore a unique center. If we can show that in all possible cases t is the center, we're done. Note that $\theta(t)=\alpha'(\alpha(t))=t$.

If $t \notin A$, this establishes that t is the center. If $t \in A$, (see fig. 19), let us show this is still true. Since $\theta(xy)=xy$ we conclude that the center of θ is on xy. Call the center u. We have $\theta(u)=\alpha'(\alpha(u))=u$. Also : $\alpha(u)=(cu)(tz)$, and $\alpha'(\alpha(u))=(c'\alpha(u))(xy)$. We conclude that $u \in c'\alpha(u)$. This is only possible if $c=c'$, which would contradict our hypothesis, or if $u=\alpha(u)=t$, which is what we wanted to show.

There are two ways the above proof could fail. One is if x, y happen to be on cc', In this case, pick $L \neq cc'$, and $b \in A$, $b \notin cc'$. Let $x'=L(xb)$, and $y'=L(yb)$. Construct θ as above, using x' and y' instead of x and y. The θ thus obtained will still take x to y. (See fig. 20).

The problem arises if $z \in A$. Then α and α' do not exist. The solution is to take $w \in xy$ such that $w \neq x$, $w \neq y$, $w \neq t$, and $w \neq z$. Then use the above proof to construct $\theta \in \Delta(t,A)$ such that $\theta(x)=w$ and $\theta' \in \Delta(t,A)$ such that $\theta'(w)=y$. The composition $\theta' o \theta$ is the collineation we need. The choice of w assumes that there are more than four points on xy. The theorem, however, is true in general, because it is a well-known result that all planes with less than ten points on a line are desarguesian. (See D p. 144).

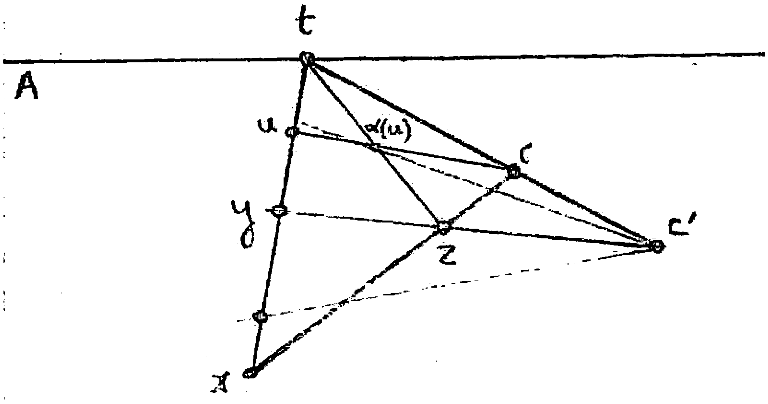


Fig 19

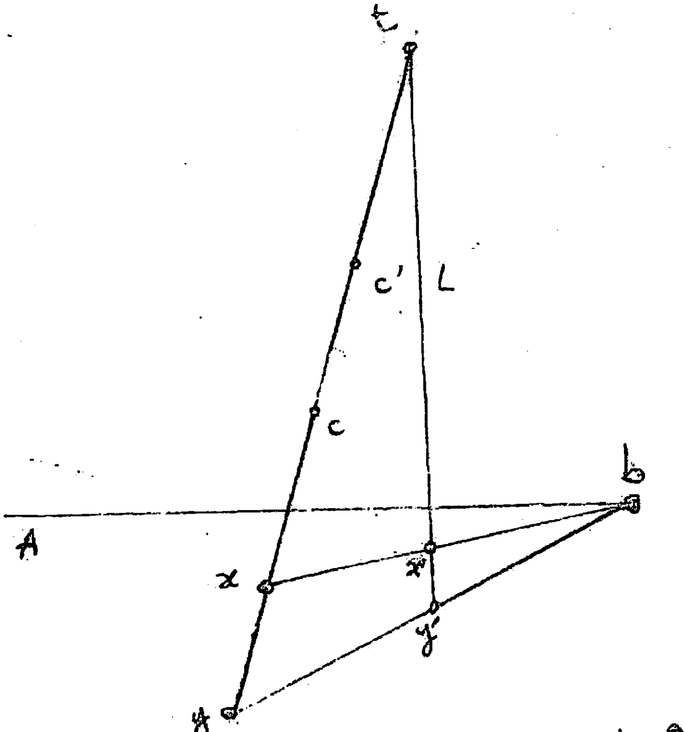


Fig 20

Case 2 : $c \in A$ and $c' \in A$. (For proof, see A p.57).

Theorem 10 : If a, b, c are three non collinear points and π is (b,ac)-transitive, and if in addition one of the following conditions holds, then π is desarguesian.

a) π is (c,ac)- and (c,ab)-transitive.

b) π is (c,A)-transitive, where A is such that $a \in A$, $b \notin A$, $c \notin A$. (See fig. 21. See B p. 216).

Proof : First notice that (a) $\Longrightarrow$ (b). Indeed, by the dual to theorem 9, (c,ac)- and (c,ab)-transitivity imply (c,a)-tansitivity, which implies condition (b). So it will be enough to show the theorem holds for (b). (For the following, see G p. 62). Let us choose o, e, x, y, no three of which are collinear, such that x=b, y=c, o=a, and $e \in A$, $e \neq (bc)A$. Then ac=oy and A=oe. So the plane is (x,oy)- and (y,oe)-transitive, so that ℓ is a (not necessarily commutative) field, (theorem 7 (g)), and hence π is desarguesian (theorem 8 (b)). End of proof of theorem 10.

Corollary : If π is (a,A)- and (b,B)-transitive, with $b \in A$, $a \notin A$, $a \notin B$, $b \notin B$, π is desarguesian.

Proof : This is just another wording of theorem 10 (b).

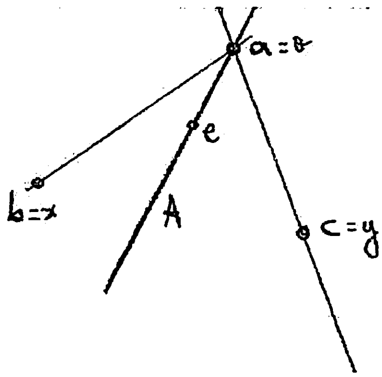


fig 21

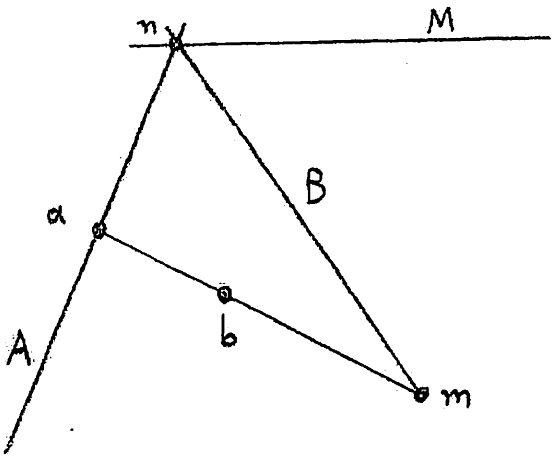


fig 22

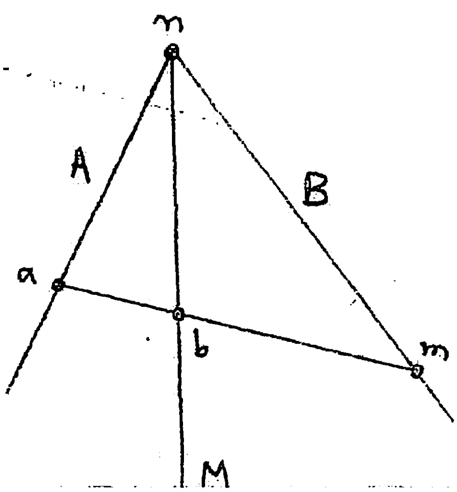


fig 23

<u>Theorem 11</u> : If π is (a,A)- and (b,B)-transitive, with $a \in A$, $a \notin B$, $b \notin A$, $b \notin B$, π is desarguesian.

<u>Proof</u> : (See <u>B</u> p. 216). Define $n=AB$, $m=(ab)B$. Consider $\alpha \in \Delta(a,A)$ such that $\alpha(b)=m$. Let $\alpha(B)=M$. Let us distinguish two cases. Case 1 : $b \notin M$ (see fig 22). By theorem 5, π is (m,M)-transitive. Since it is also (b, nm)-transitive by hypothesis we conclude, by theorem 10 (b) that π is desarguesian. Case 2 : $b \in M$. (See fig, 23). Again, by theorem 5, π is (m,M) transitive. Once more, by theorem 5, π is $(\beta(a),\beta(A))$-transitive, for all $\beta \in \Delta(b,B)$. In particular, π is (m,mn)-transitive. By theorem 10 (a) π is desarguesian. End of proof of theorem 11.

<u>Theorem 12</u> : If π is (a,b)-transitive, it is (b,a)-transitive. (See <u>D</u> p.123, <u>G</u> p. 51).

<u>Proof</u> : This is a strong result for which there seems to be no purely geometrical proof. Assume π is (a,b)-transitive. Pick o, e, x, y no three of which are collinear, such that $x=a$ and $y=b$. Construct ℓ. By theorem 7 (e), ℓ satisfies <u>I</u>,...,<u>IV</u>. Now consider the collineation α such that $\alpha(x)=y$, $\alpha((\underline{a},\underline{b}))=(\underline{b},\underline{a})$, and $\alpha((\underline{m}))=(\underline{m}^{-1})$ for $\underline{m} \neq \underline{0}$. In the proof of theorem 8 (a) we showed that α is a collineation. The proof is valid here, because <u>III</u> ==> <u>VII</u> and <u>VIII</u> and <u>VI</u>,

and the proof did not use V. We conclude that ℓ' constructed from o', e', x', y', where $o'=\alpha(o)=o$, $e'=\alpha(e)=e$, $x'=\alpha(x)=y$, and $y'=\alpha(y)=x$, is isomorphic to ℓ, and hence satisfies I,...,IV. By theorem 7 (e), π is (b,a)-transitive. End of proof of theorem 12.

Corollary : If π is (a,b)-transitive, with $a\neq b$, it is (ab,ab)-transitive.

Proof : π is in particular (a,ab)-transitive. By theorem 12, it is (b,a)-transitive. In particular, it is (b,ab)-transitive. By theorem 9, it is (ab,ab)-transitive. End of proof of corollary.

We are now fully equipped to undertake the proof of theorem 13 : the Lenz-Barlotti classification for projective planes. (Note that F will not denote a line, and φ will not denote a collineation.)

Theorem 13 : Let $F=\{(x,X)|\pi \text{ is } (x,X)\text{-transitive}\}$. Then F is of one of the following types :

I.1) $F=\emptyset$.

2) $F=\{(a,A)\}$ where $a\notin A$.

3) $F=\{(a,A),(b,B)\}$ where $a\in B$, $b\in A$, $a\notin A$, $b\notin B$.

4) $F=\{(a,A),(b,B),(c,C)\}$ where $a=BC$, $b=AC$, $c=AB$, a, b, c distinct.

5) $F=\{(x,p\varphi(x))|x\in L\}$ where $p\notin L$, and φ is a permutation of L such that $x\neq\varphi(x)$ for all x and

$\varphi o \varphi = i$.

6) $F=\{(x,\varphi(x)) \mid x \neq p,\ x \in L\}$ where $p \in L$, and φ is a bijection from $L-\{p\}$ to $\{X \mid p \in X\}-\{L\}$.

7) $F=\{(p,L)\} \cup \{(x,p\varphi(x)) \mid x \in L\}$ where p, L, φ are as in (I.5).

8) $F=\{(x,\varphi(x)) \mid x \in \pi\}$ where φ is a map from π to π^* such that $x \notin \varphi(x)$ for all x, and if $x \in \varphi(y)$ then $y \in \varphi(x)$.

II.1) $F=\{(a,A)\}$ where $a \in A$.

2) $F=\{(a,A),(b,B)\}$ where $a=AB$, $b \in A$, $a \neq b$, $A \neq B$.

3) $F=\{(p,L)\} \cup \{(x,\varphi(x)) \mid x \neq p,\ x \in L\}$ where p, L, φ are as in (I.6).

III.1) $F=\{(x,px) \mid x \in L\}$ where $p \notin L$.

2) $F=\{(p,L)\} \cup \{(x,px) \mid x \in L\}$ where $p \notin L$.

IVa.1) $F=\{(x,A) \mid x \in A\}$ for some A.

2) $F=\{(x,A) \mid x \in A\} \cup \{(b,Y) \mid b \in Y\} \cup \{(b,Z) \mid a \in Z\}$ where $a \neq b$, $a \in A$, $b \in A$.

3) $F=\{(x,X) \mid x \in A,\ \varphi(x) \in X\}$ where φ is a permutation of the points of A such that $\varphi(x) \neq x$ for all x, and $\varphi o \varphi = i$.

IVb. (1), (2) and (3) are respectively dual to IVa. (1), (2) and (3).

V.1) $F=\{(x,A) \mid x \in A\} \cup \{(b,Y) \mid b \in Y\}$ where $b \in A$.

VIa.1) $F=\{(x,X) \mid x \in A,\ x \in X\}$ for some A.

VIb.1) is the dual of VIa.1).

VII.1) $F=\{(x,X) \mid x \in X\}$ (π is a Moufang plane).

2) $F=\{(x,X) \mid \text{all } x,\ X\}$ (π is desarguesian).

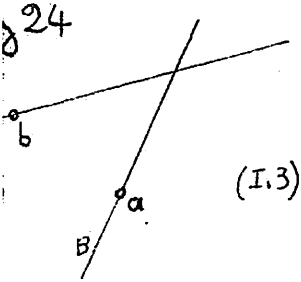
(See fig. 24. See B pp. 214-215).

Note : Lenz investigated this problem in 1954 only for (x,X)-transitivity in the case $x\in X$. He found nine types of planes (see B p.213), that he numbered I, II, III, IVa, IVb, V, VIa, VIb and VII. Barlotti studied the general case and found the above 24 types that he divided in nine classes, following Lenz's notation. It has been shown since then that planes of several Lenz-Barlotti types do not exist (see below). Dembowski defined (c,A)-transitivity for a subgroup of Δ and , following Barlotti's approach he classified these subgroups in 53 types (see D p.124-125).

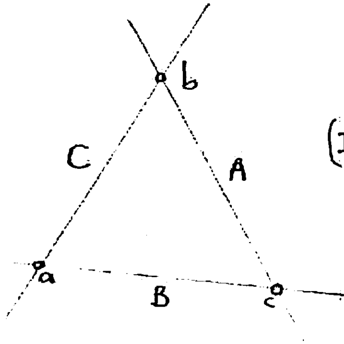
Beginning of the proof : The method of the proof is as follows (see B p. 217) : consider any plane π. Let $F(\pi)=\{(x,X)\mid\pi$ is (x,X)-transitive$\}$. Show that if $F(\pi)$ contains F (one of the figures listed in the theorem),and an additional point-line pair (q,Q), then $F(\pi)$ contains F', where F' is a figure listed below F in the theorem. (In fact, there exists an F' such that $F(\pi)=F'$. But for our proof, it will be enough to find an $F'\subset F$. Indeed, if $F'\neq F(\pi)$, then $F(\pi)$ will be discussed again later in the proof, when the case "$F(\pi)$ contains F' and an additional pair (q,Q)" is studied.)

Case 1 : F is of type (I.1). Two subcases :

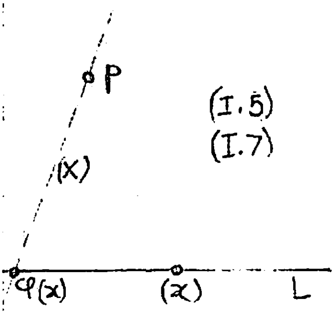
g 24



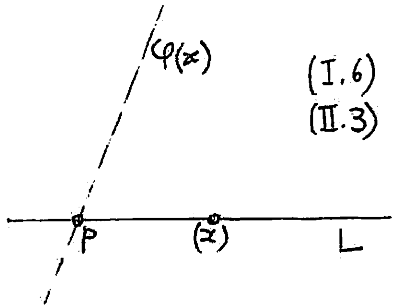
(I.3)



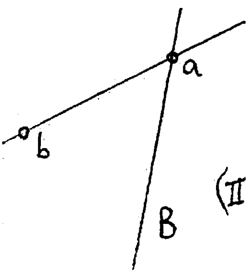
(I.4)



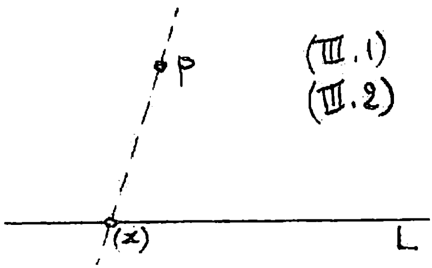
(I.5)
(I.7)



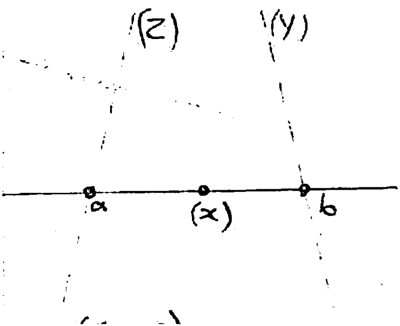
(I.6)
(II.3)



(II.2)



(III.1)
(III.2)



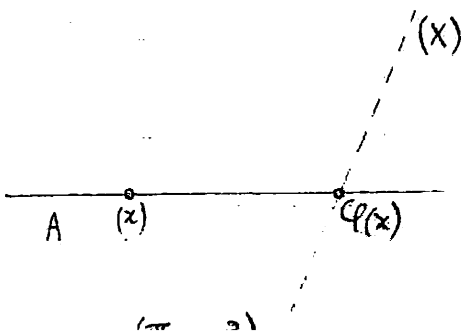
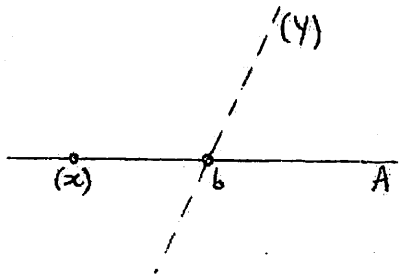
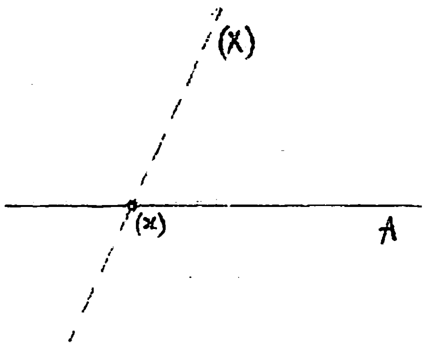


Fig 24 (cont.)

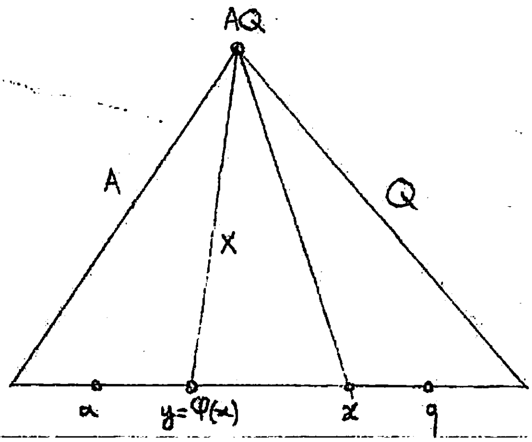


(V.1)



(VIa.1)

Fig. 25



$q \notin Q$ and $q \in Q$. They correspond respectively to F' of types (I.2) and (II.1).

Case 2 : F is of type (I.2).

Subcase : $q \neq a$, $q \notin A$, $q \notin Q$, $a \notin Q$, $A \neq Q$, $AQ \notin aq$. (See fig. 25). Then by theorem 5, using (a,A) and (q,Q) collineations, we see that all the points of aq and all the lines through AQ will belong to point-line pairs that are elements of $F(\pi)$. If one of these lines (or points) belongs to more than one pair, i.e. if $F(\pi)$ contains (x,X) and (x,X'), with $X \neq X'$, then by the dual of theorem 9, the plane is (x,AQ)-transitive. By the corollary to theorem 12, we obtain (x(AQ),x(AQ))-transitivity, and $F(\pi)$ contains F' of the type (IVa.1). The dual reasoning in the case where $F(\pi)$ contains (x,X) and (x',X), with $x \neq x'$ and x, $x' \in aq$, $AQ \in X$, yields the result that $F(\pi)$ contains F' of the type (IVb.1).

If all the lines X through AQ and points x on aq belong to one and only one pair in $F(\pi)$, define φ to be the permutation of L which associates X(aq) to x, when x and X are in the same pair. Define y=X(aq). If $x \notin (AQ)\varphi(y)$, by the corollary to theorem 10, the plane is desaguesian (i.e. $F(\pi)$ contains F' of type (VII.2)). If $x \in (AQ)\varphi(y)$ we find ourselves in the situation where $F(\pi)$ contains F' of Lenz-Barlotti type (I.5) with aq playing the role of L and AQ playing the role of p.

<u>Other subcases</u> : If $q \in Q$, we get F' of type (II.1).

If $q \notin Q$, $a \in Q$, $q \notin A$, then by the corollary to theorem 10, the plane is desarguesian. If $q \notin Q$, $a \notin Q$, $q \in A$, we get the same result by switching q and a and Q and A. (See fig. 21).

If $q \notin Q$, $a \in Q$, $q \in A$, we have F' of the type (I.3). (See fig. 24).

If $q \notin Q$, $q \notin A$, a=q, π is (a,AQ)-transitive by the dual of theorem 12, and by its corollary we obtain (a(AQ),a(AQ))-transitivity, and F' of type (IVa.1). The subcase $q \notin Q$, $a \notin Q$, Q=A is the dual of the preceding one.

Finally the last subcase is $q \notin Q$, $a \notin Q$, $q \notin A$, and $QA \in qa$. (See fig. 26). Then by theorem 5, π is (x,X)-transitive, where x is any point on aq, and X is an appropriate line, different for each x, through AQ and other than aq. We obtain F' of type (I.6).

<u>Case 3</u> : F is of type (I.3). We can avoid studying the numerous subcases by the following gimmick : Note that type (I.3) includes type (I.2) (see fig. 24). Adjoin (q,Q) to type (I.2). This was studied in case 2. Then adjoin (b,B). If adjoining (q,Q) yielded a figure F' that was after (I.3) in the theorem, we're done. The only other case is $q \notin Q$,

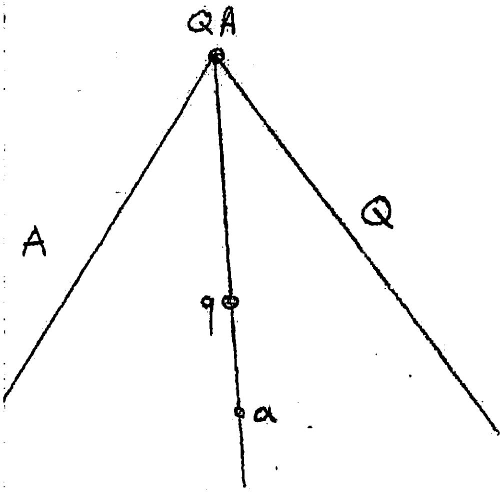


Fig 26

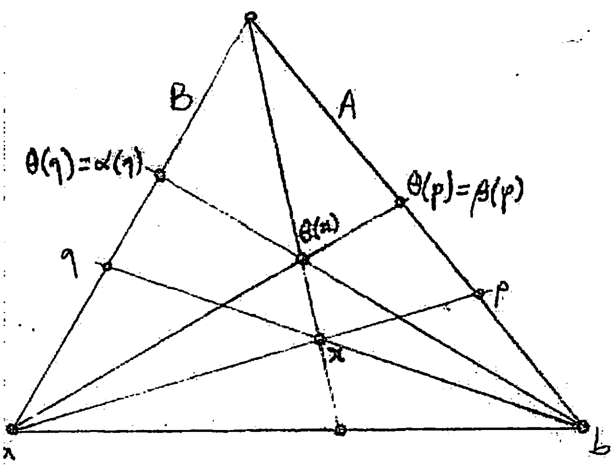


Fig 27

$a \in Q$, $q \in A$. By switching the roles of (a,A) and (b,B) we get $b \in Q$ and $q \in B$. So the only new case we have to study is $q=AB$ and $Q=ab$. This is precisely a figure of type (I.4).

The proof continues in this fashion, through 21 more cases, each one more or less complicated and more or less similar to cases 1, 2, 3. (For the remainder of the proof, see B pp.219-226.) Barlotti proved his theorem in 1957. Since then, many of the types of planes he listed were either constructed, or shown to be impossible. Several questions are still unanswered. The table on the following page represents the present state of knowledge on the question. We will comment on the non-existence of some planes in the finite case. In section(V) we will construct a number of finite non-desarguesian planes.

Theorem 14 (a) There are no projective planes of Lenz-Barlotti type(VI.1).

(b) There are no finite projective planes of type (VII.1).

Proof : (a) Consider a plane π of type (VIb.1). Then π is (X,X)-transitive for all X such that $a \in X$. Pick two distinct lines L, L' through a. Choose o, e, x, y such that $y=a$, $x \in L$, $o \in L'$, $e \notin L$, $e \notin L'$,

Lenz-Barlotti type	Plane exists (finite case)	Plane exists (infinite case)
I.1	yes	yes
I.2	?	yes
I.3	?	yes
I.4	?	yes
I.5	no	no
I.6	no	?
I.7	no	no
I.8	no	no
II.1	yes	yes
II.2	?	yes
II.3	no	no
III.1	?	yes
III.2	no	yes
IVa.1	yes	yes
IVa.2	yes	yes
IVa.3	yes	no
V.1	yes	yes
VIa.1	no	no
VII.1	no	yes
VII.2	yes	yes

For additional information about results on this table, see D p.126.

$e \notin ox$. Construct the coordinate system ℓ corresponding to o, e, x, y. By theorem 7 (f), ℓ satisfies I, II, IV, V, and VII, which implies that ℓ satisfies VI. (See E, pp.369-370, D p. 130). By theorem 8 (a), π is a Moufang plane, which contradicts our hypothesis that π is of type (VIb.1).

(b) Consider a finite plane π of type (VII.1). By theorem 8 (a), ℓ is an alternative field. But a finite alternative field is a field (see D p.130-131), and hence π is desarguesian, contradicting the hypothesis that it is of type (VII.1). This ends the proof of theorem 14. Notice that the proof was purely algebraic, based on theorem 7.

The proof of non-existence for finite planes of types (I.5), (I.6), (I.7), (I.8), (II.3) and (III.2), while requiring sophisticated algebraic results, have a significant geometric component. All of them have been developped between 1958 and 1966 (see D pp. 197-207). We will present the proof of non-existence for types (I.5) and (I.7), following D pp. 197-199 and F pp. 508-510.

Theorem 15 : There are no projective planes of Lenz-Barlotti types (I.5) and (I.7).

Lemma 4 : Let $\alpha \in \Delta(a,A)$ and $\beta \in \Delta(b,B)$, such that $\alpha o \alpha = \beta o \beta = i$, $a \in B$, $b \in A$, $A \neq B$, $a \neq b$. Then $\theta = \beta o \alpha \in \Delta(AB, ab)$ and $\theta o \theta = i$. (See D p.120).

Proof : Let $p \in A$. Since $\alpha(p)=p$, we have $\theta(p)=\beta(p)$. Similarly if $q \in B$, $\theta(q)=\alpha(q)$. Now take $x \notin ab$, $x \notin A$, $x \notin B$. Say $(bx)B=q$ and $(ax)A=p$. We must have :

$$\theta(x)=(b\alpha(q))(a\beta(p)). \text{ (See fig.27)}.$$

It follows that $\theta o \theta = i$. Now consider $(ab)(x\theta(x))$. It is clearly fixed by θ. Since it is not a or b, we conclude that ab is the axis. Since $AB \notin ab$ is also fixed by θ, it must be the center of θ. End of proof of lemma 4.

Proof of theorem : Suppose π is of type (I.5) or (I.7). Then for some point-line pair (p,L) we have : $(p \notin L)$

a) π is $(x, p\varphi(x))$-transitive for every $x \in L$, where φ is a fixed-point free permutation of L with $\varphi o \varphi = i$.

b) π is not (x,X)-transitive for any other point-line pair (x,X) except perhaps for $x=p$ and $X=L$.

Let $\underline{\Delta}$ be the permutation group induced by Δ on L. Each element $\alpha \in \Delta$ is such that $\alpha \in \Delta(x, p\varphi(x))$ or $\alpha \in \Delta(p,L)$. Indeed, say $\alpha \in \Delta$, $\alpha \notin \Delta(x,p\varphi(x))$, $\alpha \notin \Delta(p,L)$. Then $\alpha(p)=p' \neq p$ and $\alpha(L)=L' \neq L$. Therefore by theorem 5 the plane is (p',L')-transitive. Our hypothesis implies that we must have $p' \in L$, $p \in L'$, and $\varphi(p')=LL'$. It follows by theorem 5 that π is $(\alpha(x), p'\alpha(\varphi(x)))$-transitive for all $x \in L$, which contradicts the hypothesis.

To each $\alpha \in \Delta$ corresponds $\underline{\alpha} \in \underline{\Delta}$, where $\underline{\alpha}(y)=\alpha(y)$ for $y \in L$. This makes sense because elements of

$\Delta(x,p\varphi(x))$ and $\Delta(p,L)$ take points of L to points of L. Then $\varphi \circ \underline{\alpha} = \underline{\alpha} \circ \varphi$ because the images of the center and axis of an element of Δ by any collineation α will be the center and axis of some element of Δ. So φ commutes with any $\underline{\alpha} \in \underline{\Delta}$. Moreover, given any two points a and b on L, there exists a permutation $\underline{\beta} \in \underline{\Delta}$ such that $\underline{\beta}(a)=b$, since there exists $\beta \in \Delta$ such that $\alpha(a)=b$.

Consider $\underline{\alpha} \in \underline{\Delta}$ such that $\underline{\alpha}(y)=\varphi(y)$. We must have $\underline{\alpha}(\underline{\alpha}(y))=\underline{\alpha}(\varphi(y))=\varphi(\underline{\alpha}(y))=\varphi(\varphi(y))=y$. So $\underline{\alpha}$ is of order two. Assume $\underline{\alpha}$ leaves x and $\varphi(x)$ fixed, i.e. $\alpha \in \Delta(x,p\varphi(x))$ We can find $\alpha' \in \Delta(\varphi(x),px)$ with the same property $\alpha'(y)=\varphi(y)$ for $y \in L$. Lemma 4 implies that $\underline{\alpha} \circ \underline{\alpha}' = \underline{i}$. So $\underline{\alpha}=\underline{\alpha}'$. So $\underline{\alpha} \in \underline{\Delta}$ such that $\underline{\alpha}(y)=\varphi(y)$ is unique for each choice of $\{x, \varphi(x)\}$. Let us call this unique element of $\underline{\Delta}$ $\underline{\theta}[x]$. $\underline{\theta}[x]$ fixes x and $\varphi(x)$ and interchanges y and $\varphi(y)$ for $y \in L$, $y \neq x$, $y \neq \varphi(x)$.

Now introduce coordinates in π, with o=p and x, y $\in L$ such that $y=\varphi(x)$. Then we have, by theorem 7 (b): ℓ is linear and ℓ^* is isomorphic to all $\Delta(x,p\varphi(x))$. Hence ℓ^* contains precisely one element of order two, $\underline{e}$, which is in the center : $\underline{xe}=\underline{ex}$ for all $\underline{x} \in \ell$. The group $\Delta(y,ox)$ consists of the mappings $(\underline{x},\underline{y}) \longrightarrow (\underline{x},\underline{yt})$, $\underline{t} \in \ell^*$. The fact that these are collineations implies right distributivity. (This follows from an argument similar to the one in the proof of theorem 7).

The only collineation of order two in $\Delta(y,ox)$ is $\underline{\theta}[y]:(\underline{x},\underline{y})\longrightarrow(\underline{x},\underline{ye})$, mapping $(\underline{m})$ into $(\underline{me})=(\underline{em})$. Hence, if $\underline{m}\neq\underline{0}$, $\varphi((\underline{m}))=(\underline{em})=(\underline{me})$. Now consider $\underline{\theta}[(\underline{e})]$. It has center $(\underline{e})$ and axis $\underline{y}=\underline{x}$. As x and y are interchanged by $\underline{\theta}[(\underline{e})]$, it follows that :

$\underline{\theta}[(\underline{e})]((\underline{x},\underline{y}))=(\underline{y},\underline{x})$, and

$\underline{\theta}[(\underline{e})]([\underline{y}=\underline{x}+\underline{b}])=[\underline{y}=\underline{x}+(-\underline{b})]$ where $\underline{b}+(-\underline{b})=\underline{0}$.

Moreover, $(\underline{c},\underline{c}+\underline{b})$ is sent to $(\underline{c}+\underline{b},\underline{c})$, so that

$(\underline{c}+\underline{b},\underline{c})\in[\underline{y}=\underline{x}+(-\underline{b})]$. So :

$(\underline{c}+\underline{b})+(-\underline{b})=\underline{c}$, the right inversive property.

The fact that $(\underline{m},\underline{1})$ goes to $(\underline{1},\underline{m})$ implies that lines through $(\underline{m})$ where $\underline{m}\neq\underline{0}$, go into lines through $(\underline{m}^{-1})$. But $\underline{\theta}[(\underline{e})]$ must interchange $(\underline{m})$ and $\varphi((\underline{m}))=(\underline{me})$. Hence $\underline{me}=\underline{m}^{-1}$ and $\underline{m}^2=\underline{e}$ for $\underline{m}\neq\underline{1}$, $\underline{e}$, $\underline{0}$.

It follows from the right distributive law that $(-\underline{1})\underline{a}=-\underline{a}$, that is that $\underline{a}+(-\underline{1})\underline{a}=\underline{0}$ for all $\underline{a}\in\ell$. Moreover $(-\underline{a}+\underline{a})+(-\underline{a})=-\underline{a}$ and hence $-\underline{a}+\underline{a}=\underline{0}$. In particular $-\underline{e}+\underline{e}=\underline{0}$. But :

$\underline{0}=-\underline{e}+(-\underline{1})(-\underline{e})=-\underline{e}+(-\underline{1})^2\underline{e}=-\underline{e}+\underline{e}^2=-\underline{e}+\underline{1}$ (unless $\underline{e}=-\underline{1}$).

This implies that $\underline{e}=\underline{1}$. But $\underline{e}$ is of multiplicative order two, so we have a contradiction unless $\underline{e}=-\underline{1}$.

Algebraists tell us that the only $\ell^{\bullet}$ satisfying all these conditions are the cyclic groups of order 2 or 4 and the quaternion group of order 8. (See $\underline{D}$ p. 199). So ℓ has 3, 5 or 9 elements. But every projective plane of order ≤ 8 is desarguesian (see $\underline{D}$ p. 144. The order of a projective plane is the number of elements of ℓ). So ℓ^* has to be the quaternion group of order 8. Pickert showed that there are exactly two planes coordinatized by such an ℓ, and he found them to be of Lenz-Barlotti types (IVa.3) and (IVb.3) respectively, not (I.5) or (I.7). (See D p.199 for reference). This completes the proof of theorem 15 : there are no planes of Lenz-Barlotti type (I.5) or (I.7).

(V) SOME FINITE NON-DESARGUESIAN PLANES

Many constructions of such planes are known. All are outlined in D pp. 219-251. After some introductory remarks, we will describe several of them.

Planes of type (IVa) feature (A,A)-transitivity but no (b,b)-transitivity. By theorem 7 it follows that they are coordinatized by quasifields which are not semifields. There is exactly one plane of type (IVa.3), that was mentioned at the end of the proof of theorem 15. Planes of type (IVa.2) are coordinatized by planar nearfields. See D pp. 229-232, H pp. 158-159. Planes of type (IVa.1) do not have (a,b)-transitivity for any a, b. It follows by theorem 7 that they are coordinatized by quasifields that are not nearfields. See D pp.232-236, A pp.88-92. Planes of type (IV.b) are dual to the planes of type (IVa).

Planes of type (V.1) are (A,A)- and (b,b)-transitive. By theorem 7 it follows that they are coordinatized by semifields. See D pp. 236-245, A pp.86-88.

Planes of type (I.1) and (II.1) have no (A,A)- or (b,b)-transitivities. Therefore they cannot be coordinatized by a quasifield. See D pp. 246-251, C pp. 371-381, 385-387.

Here we will limit ourselves to presenting a few methods of construction, yielding planes of each of the types (IVa.2), (V.1), (IVa.1) and (I.1).

One method is to construct an algebra, and to define the plane in terms of its coordinate system. We will outline one such construction following H p158.

Let $\ell' = \{a+bj\}$ where a, b are elements of the field of three elements. Addition is defined in the natural way. Multiplication is defined using the relation $j^2=2$. It is easy to check that ℓ' is a field with nine elements. Now define ℓ as follows. $\ell=\ell'$ as a set. Addition is the same as in ℓ', but multiplication is according to the rule :

$$xy = \begin{cases} xy & \text{if } y \text{ is a square in } \ell'. \\ x^3y & \text{if not.} \end{cases}$$

Define a ternary operation in ℓ as follows :

$$x \circ m \circ b = xm+b.$$

It is not difficult to verify that with these operations ℓ is a planar nearfield and not a semifield. We saw in section (III) how a projective plane can be constructed from a ternary field ℓ. By theorem 7, the one obtained from this ℓ will be of Lenz-Barlotti type (IVa.2).

A class of planes of type (V.1) can be constructed similarly, after constructing a semi-field. (See A pp.86-88). Consider the field ℓ' of order p^n where p is a prime and $n \geq 2$ (here p and n are not points but integers). Consider $\ell = \ell' \times \ell'$ with the following operations :

$$(\underline{x},\underline{x}')+(\underline{y},\underline{y}')=(\underline{x}+\underline{y},\underline{x}'+\underline{y}')$$

$$(\underline{x},\underline{x}')(\underline{y},\underline{y}')=(\underline{x}\underline{y}+\underline{e}\underline{x}'^{p}\underline{y}',\underline{x}^{p}\underline{y}'+\underline{x}'\underline{y})$$

where $\underline{e} \in \ell'$ is fixed. It can be shown that if $\underline{e}$ is not a (p+1)st power in ℓ', then ℓ is a semi-field and not a field. The proof, unlike that of the previous example, is not a mere verification, but involves the use of the algebraic result that ℓ^{*} is cyclic and the fundamental theorem of algebra.

We will now describe an altogether different approach (see A pp. 88-91). Consider π' the plane coordinatized by the field of nine elements, ℓ'. π' is of course desarguesian. We will construct a non-desarguesian plane consisting of the same points, but different lines. Several of the lines of π actually will be lines of π'. Specifically the line xy and lines with equation $\underline{y}=\underline{x}\cdot\underline{m}+\underline{b}$, where $\underline{m} \neq \underline{0}, \underline{1}, \underline{2}$. This gives us 55 lines of π. The rest of the lines will be defined as follows. For any $\underline{x} \neq \underline{0}$, $\underline{y}$, $\underline{z}$ in ℓ' let $\underline{x}=\underline{a}\underline{j}+\underline{b}$ where $\underline{a}$, $\underline{b} \in \{\underline{0}, \underline{1}, \underline{2}\}$ and $\underline{j}^2=2$. Define $p(\underline{x})=y$ if $\underline{a}=\underline{0}$, $p(\underline{x})=(\underline{b}\underline{a}^{-1})$ if

$\underline{a} \neq \underline{0}$. Then define the set consisting of $p(\underline{x})$ and all points of the form : $(\underline{xa}+\underline{y}, \underline{xa}'+\underline{z})$ where $\underline{a}$, $\underline{a}'$ are in $\{\underline{0}, \underline{1}, \underline{2}\}$, to be a line of π, called $L(\underline{x},\underline{y},\underline{z})$. We must prove this makes π into a projective plane.

Consider the map φ that takes $(\underline{x},\underline{y})=(\underline{a}+\underline{b}\underline{j},\underline{c}+\underline{d}\underline{j})$ into $\varphi((\underline{x},\underline{y}))=(\underline{a}+\underline{c}\underline{j},\underline{b}+\underline{d}\underline{j})$, where $\underline{a}$, $\underline{b}$, $\underline{c}$, $\underline{d}$, are in $\{\underline{0}, \underline{1}, \underline{2}\}$. Note that $\varphi o \varphi = i$. It is not difficult to check that $\varphi(L(\underline{x},\underline{y},\underline{z}))$ is a line of π' with equation $\underline{y}=\underline{xm}+\underline{b}$ where $\underline{m} \in \{\underline{0}, \underline{1}, \underline{2}\}$, and φ is a bijection. It follows that we defined the right number of lines $L(\underline{x},\underline{y},\underline{z})$. The axioms PI, PII, PIII are easily proved, using the bijection φ.

A legitimate question is : how do we know π is not isomorphic to π' ? This can be shown by proving that the plane is not $(y,L(\underline{j},\underline{0},\underline{0}))$-desarguesian. (Consider the triangles $(\underline{j}+\underline{1})$, $(\underline{j},\underline{j}+\underline{1})(\underline{2j}+\underline{1},\underline{0})$ and $(\underline{j})(\underline{j},\underline{1})(\underline{1},\underline{2j})$).

On page 92, $\underline{A}$ claim that π can be coordinatized by a quasifield that is not a nearfield or a semifield. This makes it a plane of type (IVa.1).

We will now construct a plane of type (I.1), known as a Hughes plane, (named after Hughes who discovered it in 1957). We will basically follow $\underline{C}$ pp. 379-381 and 385-387. However, since Hughes uses rather unusual definitions, our presentation will be somewhat different.

Definition : A (left) nearfield is an algebraic system satisfying II, III, and V. (See section (III)).

Zassenhaus has shown that for any odd prime p and any positive integer n, there is a nearfield of order p^{2n} which is not a field, but whose center is a field of order p^n. Call such a nearfield N and its center F. (From here on lower case letters do not necessarily denote points, and capitals do not necessarily denote lines.)

Let $q=p^{2n}+p^n+1$. Let $V=N\times N\times N$, and $V'=F\times F\times F$. Then V is a left vector space over N and V' is a left vector space over F. Let φ be a non-singular linear transformation such that

a) $\varphi(V')=V'$

b) If $v\in V$, $\varphi^q(v)=kv$ for some $k\in N$, $k\neq 0$.

c) If $v'\in V'$, $v'\neq(0,0,0)$, and $\varphi^m(v')=kv'$ where $k\in F$, $k\neq 0$, then $m\equiv 0 \pmod q$.

For proof of the existence of such a φ, see reference in C p. 380.

The points of π will be the elements of V, except (0,0,0), where (x,y,z) and (kx,ky,kz) are identified. The lines are $\varphi^m(L(t))$, where $t\in N$, with t=1 or $t\notin F$. $\varphi^k(L(t))$ and $\varphi^m(L(t))$ are identified if and only if $k\equiv m \pmod q$. We have $(x,y,z)\in L(t)$ if and only if $x-y+zt=0$. π contains $p^{4n}+p^{2n}+1$ points and the same number of lines, with $p^{2n}+1$ points on each line. By simple computations we can show that π satisfies the axioms for a projective plane. π is called a Hughes plane and it has a desarguesian subplane π' consisting exactly of the points (x,y,z) for which x, y, z are in F, and the lines $\varphi^m(L(1))$. Note that a line $\varphi^m(L(t))$ can be represented by an equation : $xa+yb+zc+(xa'+yb'+zc')t=0$, where a, b, c, a', b', c' are in F.

Now let us coordinatize π in the usual fashion. Let $x=(1,0,0)$, $y=(0,1,0)$, $o=(0,0,1)$ and $e=(1,1,1)$. Then the lines ox, oy, xy, oe are lines of the form $\varphi^k(L(1))$, and in particular they have equations respectively $y=0$, $x=0$, $z=0$ and $x=y$. Therefore the old coordinates of $(oe)(xy)=(1)$ are $(1,1,0)$. We can see that all the lines through (1) are of the form $x-y+zt=0$, for $t\in N$.

The coordinates of the points of oe are of the form $(v,v,1)$ in the old system, where $v\in N$. Rename them $(\underline{v},\underline{v})$. What are the new coordinates of points of ox ? Consider the line through $y=(0,1,0)$ and $(u,0,1)$, a point of ox. Where does it intersect $x=y$? If it has equation :

$$xa+yb+zc+(xa'+yb'+zc')t=0,$$

then we have : $b+b't=0$ and $ua+c+(ua'+c')t=0$. Is the point $(u,u,1)$ on the line ? We consider two cases. If $t=1$, then clearly yes. If $t\notin F$, since b and b' are in F, we must have b=b'=0. In this case again (u,u,1)=(u,u) is on the line. So (u,0,1) can be renamed (u,0). Similarly for points of oy (0,u,1)=(0,u).

What are the old coordinates of (u,v) ? (u,v) is the intersection of y(u,0) and x(0,v). Say the equations of these lines are respectively :

$$xa+yb+zc+(xa'+yb'+zc')t=0,$$

$$xd+ye+zf+(xd'+ye'+zf')s=0.$$

Then we have :

$$b+b't=ua+c+(ua'+c')t=0,$$

$$d+d's=ve+f+(ve'+f')s=0.$$

As before, either t=1 and b+b'=0 or t∉F and b=b'=0. Similarly, either s=1 and d+d'=0 or s∉F and d=d'=0. This gives us four cases. It is easy to check that in all four cases (u,v,1) is on both lines. We conclude (u,v)=(u,v,1).

(m) is on (1,m)o and xy. xy is z=0. Say (1,m)o has equation : xa+yb+zc+(xa'+yb'+zc')t=0, then we have : c+c't=a+mb+c+(a'+mb'+c')t=0. In both cases (t71 and t∉F) it is clear that (1,m,0) is on this line. We conclude :

Lemma 5 : (u,v) is (u,v,1) ; (m) is (1,m,0) ; (∞) is (0,1,0).

Now we will investigate ℓ, the ternary field defined as in section (III) from the operation

v=u.mob.

Lemma 6 : $\underline{x}.\underline{1}o\underline{v}=\underline{x}+\underline{v}$, $\underline{x}.\underline{m}o\underline{0}=\underline{xm}$.

Proof : The line through ($\underline{1}$) and ($\underline{0},\underline{v}$) is the line $\underline{y}=\underline{x}.\underline{1}o\underline{v}$. We must show that the point ($\underline{x},\underline{x+v}$) is on it. Say its equation is $x-y+zt=0$, then clearly $t=v$, and $(x,x+v,1)$ is on it.

The line through ($\underline{m}$) and o has equation $\underline{y}=\underline{x}.\underline{m}o\underline{0}$. We must show ($\underline{x},\underline{xm}$) is on the line. Well if its equation in the old system was :

$$xa+yb+zc+(xa'+yb'+zc')t=0,$$

we have : $c+c't=a+mb+(a'+mb')t=0$. In both cases ($t=1$ and $t\notin F$) we see that $(x,xm,1)$ is on the line. End of proof of lemma 6.

Lemma 7 : ℓ is not linear.

Proof : Lemma 6 showed that ℓ is isomorphic to N. Therefore it satisfies II, III, and V of section (III). Assume ℓ is linear (satisfies I). Then π is coordinatized by a planr nearfield. Hence by theorem 7, there exists a point $p\in\pi$ such that π is (p,p)-transitive. But φ is a collineation of order $q=p^{2n}+p^{n}+1$ which leaves no point fixed. So the plane is $(\varphi(p),\varphi(p))$- and $(\varphi^2(p),\varphi^2(p))$-transitive (theorem 5). Since $q>3$, $\varphi^2(p)\neq p$. So by the dual of the corollary to lemma 3, this implies that the plane is Moufang, and since it is finite it must be desarguesian. So ℓ and hence N is a field, a contradiction. End of proof of lemma7.

<u>Theorem 16</u> : A Hughes plane is of Lenz-Barlotti type (I.1).

<u>Proof</u> : We do not have the necessary machinery to prove this theorem. (See <u>D</u> p.248). It is clear however that π is not (A,A)- or (b,b)-transitive for any line A or point b. So π cannot be coordinatized by a quasifield, and therefor must be of Lenz-Barlotti type (I.1) or (II.1).

The Hughes planes are the "least" desarguesian possible. It is amusing to note, however, that they, like all finite projective planescontain many Desargues configurations such as the one of fig. 1. (See <u>D</u> p. 145).

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